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RENOMALIZED NOTES FOR NAVIER-STOKES EQUATIONS BY ROGER TEMAM

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ABSTRACT. In this paper, we give a renormalized note for Roger Temam's book [3], sometimes I omit trivialities, while other times I add comments—giving more direct proof, totally due to the author's interest and limited knowledge.

CONTENTS

Key words and phrases. Navier-Stokes equations.

Part 0. Preface

As is well-known (see [1]), the issue of

1. regularity and uniqueness of weak solutions,
2. global existence of strong solutions,

of the Navier-Stokes equations

$$\left.\begin{array}{c}\partial_t \boldsymbol{u} - \nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} = 0\end{array}\right\} \text{ in } \boldsymbol{R}^n\times(0,T), \qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \text{in } \boldsymbol{R}^n. \tag{1}$$

is one of the seven millennium prize problems. To handle this, a lot of mathematicians devoted their valuable time in thinking, experimenting, etc.

The book [3] by Temam is a fundamental one, and here I renormalize it from a pure mathematical point of view, omitting all of those numerical results.

Let us describe briefly what the tedious job contains.

Chapter 1 deals with the linear, stationary version of (1), namely, deleting the convective term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$, neglecting the evolutionary term $\partial_t\boldsymbol{u}$ in (1).

Chapter 2 concerns about the stationary version of (1), that is, omitting only $\partial_t\boldsymbol{u}$ in (1).

Chapter 3 is a almost complete treatment of classical results of (1). Precisely,

1. global existence of a weak solution;
2. unique solvability when $n=2$;
3. unique solvability when $n=3$, under suitable "smallness" data;
4. local existence of strong solutions;
5. decay of solutions for $n=2,3$.

At this time , I would like to express my sincere gratitudes to the following

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from whom (the people) and which (the discussion group), I learned and suffered a lot.

Finally, God blesses my family! In particular my parents, for their hard working and the job of raising me with no gain up to now!

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I've known swimming a little bit.
Happy the National Day.

Zujin Zhang

Part 1. The steady-state Stokes equations

1. Some function spaces

1.1. Notation

1. The set $\Omega(\subset \boldsymbol{R}^n)$ with boundary Γ and outward unit normal $\boldsymbol{\nu}$.

$$C^r(r \geq 1) \rightsquigarrow \text{locally Lipschitz} \rightsquigarrow \text{locally star-shaped}.$$

   We shall always assume Ω is locally Lipschitz, unless otherwise stated.

2. L^p and Sobolev spaces.

$L^p(\Omega)$ and its vector analogy $\boldsymbol{L}^p(\Omega)$;

$W^{m,p}(\Omega)$ and its vector analogy $\boldsymbol{W}^p(\Omega)$;

$\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(\bar{\Omega})$) $=$ $\{C^\infty$ functions with compact support in Ω $(resp.\bar{\Omega})\}$ and its vector analogy $\boldsymbol{\mathcal{D}}(\Omega)$ (resp$\boldsymbol{\mathcal{D}}(\bar{\Omega})$);

$\boldsymbol{\mathcal{V}} = \{\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega);\ \text{div}\ \boldsymbol{u} = 0\}$;

$\boldsymbol{H}$ = completion of $\boldsymbol{\mathcal{V}}$ under $\|\cdot\|_2$;

$\boldsymbol{V}$ = completion of $\boldsymbol{\mathcal{V}}$ under $\|\cdot\|_{1,2}$.

1.2. A density theorem

Denote by

$$\boldsymbol{E}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega);\ \text{div}\ \boldsymbol{u} \in L^2(\Omega)\right\}.$$

This is a Hilbert space with the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{E}(\Omega)} = (\boldsymbol{u}, \boldsymbol{v}) + (\text{div}\ \boldsymbol{u}, \text{div}\ \boldsymbol{v}).$$

Theorem 1. *Let Ω be a Lipschitz open set in $\boldsymbol{R}^n$. Then the set of vector functions belonging to $\boldsymbol{\mathcal{D}}(\bar{\Omega})$ is dense in $\boldsymbol{E}(\Omega)$.*

Sketch of Proof of Theorem 1

1. Approximate by functions with compact support in $\bar{\Omega}$.
2. By partition of unity, we may assume w.l.g. that $\Omega = \boldsymbol{R}^n$.
3. Approximate then by functions in $\boldsymbol{\mathcal{D}}(\bar{\Omega})$ through regularization.

□

1.3. A trace theorem

For an open, bounded set Ω of class C^2, it is well-known that

$$H^1(\Omega) \leftrightsquigarrow H^{1/2}(\Gamma) \subset L^2(\Gamma).$$

We show an analogous result for $\boldsymbol{E}(\Omega)$, that is,

Theorem 2. *Let Ω be an open, bounded set of class C^2. Then there exists a linear continuous operator $\gamma_{\boldsymbol{\nu}} : \boldsymbol{E}(\Omega) \to H^{-1/2}(\Gamma)$, such that*

$$\gamma_{\boldsymbol{\nu}} \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nu}|_{\Gamma}, \ \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\bar{\Omega}).$$

Moreover, we have the generalized Stokes formula:

$$(\boldsymbol{u}, \nabla w) + (div\ \boldsymbol{u}, w) = \langle \gamma_{\boldsymbol{\nu}} \boldsymbol{u}, w|_{\Gamma} \rangle, \ \boldsymbol{u} \in \boldsymbol{E}(\Omega), \ w \in H^1(\Omega). \tag{2}$$

Sketch of Proof of Theorem 2

1. For $\boldsymbol{u} \in \boldsymbol{E}(\Omega)$, define a map

$$X_{\boldsymbol{u}} : H^{1/2}(\Gamma) \ni \phi \mapsto (\boldsymbol{u}, \nabla w) + (\text{div}\ \boldsymbol{u}, w),$$

   where $w|_{\Gamma} = \phi$.
2. Show that $X_{\boldsymbol{u}}(\phi)$ is well-defined, i.e. it is not independent of w.
3. Prove that $X_{\boldsymbol{u}}$ is bounded and linear, so that by Riesz representation theorem,

$$\exists\, g \in H^{-1/2}(\Gamma), \ s.t.\ X_{\boldsymbol{u}}(\phi) = \langle g, \phi \rangle, \ \phi \in H^{1/2}(\Gamma).$$

4. Verify then

$$\gamma_{\boldsymbol{\nu}} \boldsymbol{u} \equiv g = \boldsymbol{u} \cdot \boldsymbol{\nu}|_{\gamma}, \ \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\bar{\Omega}).$$

□

Remark 3. *The trace mapping*

$$\gamma_{\boldsymbol{\nu}} : \boldsymbol{E}(\Omega) \to H^{-1/2}(\Gamma)$$

is onto.

Proof

1. *For $\phi \in H^{-1/2}(\Gamma)$ with $\langle \phi, 1 \rangle = 1$, consider*

$$\begin{cases} \Delta p = 0, & in\ \Omega; \\ \frac{\partial p}{\partial \boldsymbol{\nu}} = \phi, & on\ \Gamma. \end{cases}$$

Then $p \in H^1(\Omega)$, which is unique up to a constant. Denote by $\boldsymbol{u} = \nabla p$, then we have $\gamma_{\boldsymbol{\nu}} \boldsymbol{u} = \phi$.

2. *For general $\psi \in H^{-1/2}(\mathit{\Gamma})$, we choose a C^1 $\boldsymbol{u}_0 \in \boldsymbol{E}(\Omega)$ such that $\gamma_{\boldsymbol{\nu}} \boldsymbol{u}_0 = 1$. Then the equality*

$$\begin{aligned} \psi &= \left(\psi - \frac{\langle \psi, 1\rangle}{|\mathit{\Gamma}|}\right) + \frac{\langle \psi, 1\rangle}{|\mathit{\Gamma}|} \\ &= \phi + \frac{\langle \psi, 1\rangle}{|\mathit{\Gamma}|}, \end{aligned}$$

yields an

$$\boldsymbol{u} = \nabla p + \frac{\langle \psi, 1\rangle}{|\mathit{\Gamma}|} \boldsymbol{u}_0,$$

such that $\gamma_{\boldsymbol{\nu}} \boldsymbol{u} = \psi$.

□

Theorem 4. *The kernel of $\gamma_{\boldsymbol{\nu}}$ is equal to $\boldsymbol{E}_0(\Omega)$, which is the completion of $\boldsymbol{\mathcal{D}}(\Omega)$ under $\|\cdot\|_{\boldsymbol{E}(\Omega)}$.*

Remark 5. *If Ω is unbounded or $\mathit{\Gamma}$ is not smooth, then only partial results hold true. For example,*

1. *if $\mathit{\Gamma}_0(\subset \mathit{\Gamma})$ is bounded of C^2, then $\gamma_{\boldsymbol{\nu}} \boldsymbol{u}$ is defined $on \mathit{\Gamma}_0$, and $\gamma_{\boldsymbol{\nu}} \boldsymbol{u} \in H^{-1/2}(\mathit{\Gamma}_0)$;*
2. *if Ω is smooth but unbounded or if $\mathit{\Gamma}$ is the union of a finite number of bounded $(n-1)$ dimensional manifolds of C^2, the $\gamma_{\boldsymbol{\nu}} \boldsymbol{u}$ is also defined.*

Nevertheless, in both cases, the generalized Stokes formula (2) does not hold.

Remark 6. *Theorems 1, 2,4 hold for a more general domain Ω, for example if Ω satisfies the following two conditions:*

1. $$H^1(\Omega) \leftrightsquigarrow H^{1/2}(\Omega);$$
2. *Ω is Lipschitz.*

1.4. **Characterization of the spaces H and V**

Recall that

$$\boldsymbol{\mathcal{V}} = \{\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega);\ \text{div}\ \boldsymbol{u} = 0\},$$

$$\boldsymbol{H} = \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in } L^2(\Omega),$$

$$\boldsymbol{V} = \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in } H_0^1(\Omega).$$

1. Characterization of the gradient of a distribution.

   Let Ω be an open set in $\boldsymbol{R}^n$, and $p \in \boldsymbol{\mathcal{D}}'(\Omega)$, the space of distributions, then it obvious that

$$\langle \nabla p, \boldsymbol{v} \rangle = -\langle p, \text{div}\ \boldsymbol{v} \rangle = 0,\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}.$$

   Moreover, the converse of this property is also true. In fact we have

   Proposition 7. *Let Ω be an open set of $\boldsymbol{R}^n$, and $\boldsymbol{f} = (f_1, \cdots, f_n)$, $f_i \in \boldsymbol{\mathcal{D}}'(\Omega)$, $i = 1, \cdots, n$. A necessary and sufficient condition that*

$$\boldsymbol{f} = \nabla p,$$

   for some $p \in \boldsymbol{\mathcal{D}}'(\Omega)$, is that

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = 0,\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}.$$

   Proof. This is just a restatement of a result of de Rham, and it is well-known in differential geometry. □

   Proposition 8. *Let Ω be a bounded Lipschitz open set in $\boldsymbol{R}^n$, and $p \in \boldsymbol{\mathcal{D}}'(\Omega)$.*
   (a) If $\nabla p \in \boldsymbol{L}^2(\Omega)$, then $p \in L^2(\Omega)$, and

$$\|p\|_{L^2(\Omega)/\boldsymbol{R}} \le c(\Omega)\, \|\nabla p\|_{\boldsymbol{L}^2(\Omega)}. \tag{3}$$

   (b) If $\nabla p \in \boldsymbol{H}^{-1}(\Omega)$, then $p \in L^2(\Omega)$, and

$$\|p\|_{L^2(\Omega)/\boldsymbol{R}} \le c(\Omega)\, \|\nabla p\|_{\boldsymbol{H}^{-1}(\Omega)}. \tag{4}$$

In both cases, if Ω is any open set in $\boldsymbol{R}^n$, we have $p \in L^2_{loc}(\Omega)$.

Remark 9. *(a) Combining the results of Propositions 7 and 8, we see that*

$$\left.\begin{array}{l} \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)\,(\textit{resp. } \boldsymbol{L}^2_{loc}(\Omega)) \\ \langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0,\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}} \end{array}\right\} \Rightarrow \boldsymbol{f} = \nabla p \textit{ with } p \in L^2_{loc}(\Omega).$$

If moreover, Ω is Lipschitz and bounded, then $p \in L^2(\Omega)($ resp. $H^1(\Omega))$.

(b) (4) implies that ∇ is an isomorphism form $L^2(\Omega)/\boldsymbol{R}$ into $\boldsymbol{H}^{-1}(\Omega)$; hence the range of this linear operator is closed.

(c) Recall that if Ω is bounded,

$$L^2(\Omega)/\boldsymbol{R} = \left\{ p \in L^2(\Omega);\ \int_\Omega p = 0 \right\}.$$

2. Characterization of the space $\boldsymbol{H}$.

Theorem 10. *Let Ω be a Lipschitz open bounded set in $\boldsymbol{R}^n$. Then*

$$\boldsymbol{H}^\perp = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega); \boldsymbol{u} = \nabla p,\ p \in H^1(\Omega) \right\},$$

$$\boldsymbol{H} = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \gamma_{\boldsymbol{\nu}} \boldsymbol{u} = 0 \right\}.$$

Remark 11. *(a) If Ω is any open set in $\boldsymbol{R}^n$, then*

$$\boldsymbol{H}^\perp = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{u} = \nabla p,\ p \in L^2_{loc}(\Omega) \right\}.$$

(b) If Ω is unbounded an locally Lipschitz, then

$$\boldsymbol{H}^\perp = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{u} = \nabla p,\ p \in L^2_{loc}(\bar{\Omega}) \right\}.$$

Theorem 12. *Let Ω be an open bounded set of class C^2. Then*

$$\boldsymbol{L}^2(\Omega) = \boldsymbol{H} \oplus \boldsymbol{H}_1 \oplus \boldsymbol{H}_2,$$

where

$$\boldsymbol{H}_1 = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \boldsymbol{u} = \nabla p,\ p \in H^1(\Omega),\ \Delta p = 0 \right\},$$

$$\boldsymbol{H}_2 = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \boldsymbol{u} = \nabla p,\ p \in H^1_0(\Omega) \right\}.$$

Proof. For $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, consider

$$\begin{cases} \Delta p = \operatorname{div} \boldsymbol{u}, & \text{in } \Omega, \\ p = 0, & \text{on } \mathit{\Gamma}, \end{cases}$$

and let $\boldsymbol{u}_2 = \nabla p \in \boldsymbol{H}_2(\Omega)$.

Then we consider

$$\begin{cases} \Delta q = 0, & \text{in } \Omega, \\ \frac{\partial q}{\partial \boldsymbol{\nu}} = \gamma_{\boldsymbol{\nu}} \left(\boldsymbol{u} - \nabla p \right), & \text{on } \mathit{\Gamma}, \end{cases}$$

and let

$$u_1 = \nabla q \in \boldsymbol{H}_1(\Omega),$$

$$u_0 = u - u_1 - u_2 \in \boldsymbol{H}(\Omega).$$

□

Remark 13. *Denote by $P_{\boldsymbol{H}}$ the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}(\subset \boldsymbol{L}^2(\Omega))$, then if $\Omega \in C^{r+1}(r \geq 1)$, then*

$$P_{\boldsymbol{H}} : \boldsymbol{H}^r(\Omega) \to \boldsymbol{H}^r(\Omega).$$

3. Characterization of the space $\boldsymbol{V}$.

Theorem 14. *Let Ω be an open Lipschitz bounded set. Then*

$$\boldsymbol{V} = \left\{ \boldsymbol{u} \in H_0^1(\Omega);\ div\, \boldsymbol{u} = 0 \right\}.$$

Remark 15. *A result weaker than Propositions 7 and 8 says that*

$$\left. \begin{array}{l} \Omega \textit{ Lipschitz, bounded and open} \\ \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega),\ \langle \boldsymbol{f}, \boldsymbol{v} \rangle = 0,\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}} \end{array} \right\} \Rightarrow \boldsymbol{f} = \nabla p,\ p \in L^2(\Omega),$$

$$\left. \begin{array}{l} \Omega \textit{ open} \\ \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega),\ \langle \boldsymbol{f}, \boldsymbol{v} \rangle = 0,\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}} \end{array} \right\} \Rightarrow \boldsymbol{f} = \nabla p,\ p \in L^2_{loc}(\Omega).$$

2. Existence and uniqueness for the Stokes equations

2.1. Variational formulation of the problem

1. The problem we consider is

$$\left.\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \text{div } \boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \boldsymbol{u} = 0, \quad \text{on } \Gamma. \tag{1}$$

Here $\nu > 0$ is the kinematic viscosity, $\boldsymbol{u} = (u_1, \cdots, u_n)$ is the velocity filed, p is a scalar pressure, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is the external force, Ω with boundary Γ is open, bounded in $\boldsymbol{R}^n$.

2. Variational formulation of (1).

Definition 16. *A measurable vector $\boldsymbol{u}$ is said to be a weak solution to* (1) *iff*

(a) $\boldsymbol{u} \in \boldsymbol{V}$*;*

(b) the following equality holds:

$$\nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right), \ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}. \tag{2}$$

Remark 17. *(a) The weak solution was introduced by J. Leray.*

(b) Once $\boldsymbol{u}$ is found, we can associate a p such that

$$-\nu\Delta \boldsymbol{u} - \boldsymbol{f} = -\nabla p,$$

by Proposition 7. Of course, $p \in L^2_{loc}(\Omega)$ for general open Ω, but $p \in L^2(\Omega)$ if Ω is bounded, open and Lipschitz.

(c) Due to the open problem that whether or not $\boldsymbol{V}$ equals to

$$\boldsymbol{V}^\bullet \equiv \left\{\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega);\ div\ \boldsymbol{u} = 0\right\},$$

we can however gave two types of different variational formulations, i.e. $\boldsymbol{V}$ may be replaced by $\boldsymbol{V}^\bullet$ in Definition 16. For technical reasons, we shall consider only the $\boldsymbol{V}-$case.

2.2. The projection theorem

1. Existence of a unique weak solution.

Theorem 18. *For any open set $\Omega \subset \boldsymbol{R}^n$ which is bounded in some direction, the problem (1) has an unique weak solution $\boldsymbol{u}$. Moreover, we can associate a*

$$p \in \begin{cases} L^2_{loc}(\Omega), \\ L^2(\Omega), & \text{if } \Omega \text{ is bounded of class } C^2, \end{cases}$$

such that

$$-\Delta \boldsymbol{u} - \boldsymbol{f} = -\nabla p, \text{ in } \boldsymbol{\mathcal{D}}'(\Omega).$$

Proof. We need only establish the existence of an unique weak solution to (1), while which is a simple consequence of the following classical projection theorem. □

2. A projection theorem.

Theorem 19. *Let $\boldsymbol{W}$ be a separable real Hilbert space with norm $\|\cdot\|_{\boldsymbol{W}}$, and $a(\boldsymbol{u}, \boldsymbol{v})$ is a bilinear continuous form on $\boldsymbol{W} \times \boldsymbol{W}$, which is coercive, i.e.*

$$\exists\, \alpha > 0, \ s.t.\ a(\boldsymbol{u}, \boldsymbol{u}) \geq \alpha \|\boldsymbol{u}\|^2_{\boldsymbol{W}}, \ \boldsymbol{u} \in \boldsymbol{W}.$$

Then for each $\boldsymbol{l} \in \boldsymbol{W}'$,

$$\exists \mid \boldsymbol{u} \in \boldsymbol{W}, \ s.t.\ \langle \boldsymbol{l}, \boldsymbol{v} \rangle = a(\boldsymbol{u}, \boldsymbol{v}), \ \boldsymbol{v} \in \boldsymbol{W}.$$

Proof. (a) Uniqueness.

(b) Existence.

We just use a Galerkin approximation method.

□

3. A variation property.

Proposition 20. *The weak solution to* (1) *is also the unique element in* $\boldsymbol{V}$ *such that*

$$E(\boldsymbol{u}) \leq E(\boldsymbol{v}),\ \boldsymbol{v} \in \boldsymbol{V},$$

where

$$E(\boldsymbol{v}) = \frac{\nu}{2}\left\|\nabla \boldsymbol{v}\right\|_2^2 - (\boldsymbol{f}, \boldsymbol{v}).$$

Remark 21. *If* $\boldsymbol{V}$ *and* $\boldsymbol{V}^{\bullet}$ *are different, Theorem 19 also applies to establish an unique weak solution* $\tilde{\boldsymbol{u}}$ *of* (1) *with* $\boldsymbol{V}$ *replaced by* $\boldsymbol{V}^{\bullet}$*. Also Proposition 20 holds with* $(\boldsymbol{u}, \boldsymbol{V})$ *replaced by* $(\tilde{\boldsymbol{u}}, \boldsymbol{V}^{\bullet})$*.*

2.3. **The unbounded case**

Here we consider the case when Ω is unbounded in all directions. The difficulty resides in the invalidation of Poincaré inequality, which is to ensure that for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the map

$$\boldsymbol{V} \ni \boldsymbol{v} \mapsto (\boldsymbol{f}, \boldsymbol{v}) \in \boldsymbol{R},$$

is bounded and linear.

To overcome this difficulty, we introduce

$$\boldsymbol{Y} = \text{the completion of } \boldsymbol{\mathcal{V}} \text{ under } \left\|\nabla \cdot\right\|_2,$$

and consider $\boldsymbol{f} \in \boldsymbol{Y}'$. Precisely, we have

Theorem 22. *Let* Ω *be an open set in* $\boldsymbol{R}^n$*, and let* $\boldsymbol{f} \in \boldsymbol{Y}'$*. Then*

$$\exists \mid \boldsymbol{u} \in \boldsymbol{Y},\ s.t.\ \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\ \boldsymbol{v} \in \boldsymbol{Y}.$$

Moreover, we can associate a

$$p \in \begin{cases} L^2_{loc}(\Omega), \\ L^2_{loc}(\bar{\Omega}), & \text{if } \Omega \text{ is locally Lipschitz}, \end{cases}$$

such that

$$-\nu \Delta \boldsymbol{u} - \boldsymbol{f} = -\nabla p,\ \text{ in } \boldsymbol{\mathcal{D}}'(\Omega).$$

Remark 23. *Due to the following*

Lemma 24. $\boldsymbol{Y} \subset \{\boldsymbol{u} \in \boldsymbol{L}^{\alpha}(\Omega);\ \nabla \boldsymbol{u} \in \boldsymbol{L}^{2}(\Omega)\}$, *with* $\alpha = \dfrac{2n}{n-2}$, *if* $n \geq 3$, *and* $Y \subset \{\boldsymbol{u} \in \boldsymbol{L}^{\alpha}(\Omega);\ \nabla \boldsymbol{u} \in \boldsymbol{L}^{2}(\Omega)\}$, $\alpha \geq 1$, *for* $n = 2$. *The injections are continuous.*

We may take $\boldsymbol{f} \in L^{\alpha'}(\Omega)$ *with* $1/\alpha + 1/\alpha' = 1$ *in Theorem 22.*

2.4. The non-homogeneous Stokes equations

We consider here a non-homogeneous Stokes problem:

$$\left.\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \text{div } \boldsymbol{u} = g \end{aligned}\right\} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{\phi} \quad \text{on } \mathit{\Gamma}. \tag{3}$$

We have the following result.

Theorem 25. *Let* Ω *be an open bounded set of class* C^2 *in* $\boldsymbol{R}^n$, *and* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}^{1/2}(\Omega)$ *such that*

$$\int_{\Omega} g dx = \int_{\mathit{\Gamma}} \boldsymbol{\phi} \cdot \boldsymbol{\nu} d\mathit{\Gamma}. \tag{4}$$

Then

$$\exists\, \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ p \in L^2(\Omega)$$

which satisfies (3).

Moreover, $\boldsymbol{u}$ *is unique and* p *is unique up to a constant.*

Proof. The fact that $\boldsymbol{\phi} \in \boldsymbol{H}^{1/2}(\mathit{\Gamma})$ implies

$$\exists\, \boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega),\ s.t.\ \boldsymbol{u}_0|_{\mathit{\Gamma}} = \boldsymbol{\phi}.$$

Moreover,

$$\int_{\Omega} \text{div } \boldsymbol{u}_0 dx = \int_{\mathit{\Gamma}} \boldsymbol{\phi} \cdot \boldsymbol{\nu} d\mathit{\Gamma} = \int_{\Omega} g dx.$$

Due to the following

Lemma 26. *Let Ω be a Lipschitz open bounded set in $\boldsymbol{R}^n$. Then div : $\boldsymbol{H}_0^1(\Omega) \to L^2(\Omega)/\boldsymbol{R}$ is onto.*

Proof of the Lemma Noticing that

$$\nabla : L^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega),$$

is isomorphic from

$$L^2(\Omega)/\text{Ker }(\nabla) = L^2(\Omega)/\boldsymbol{R} \to R(\nabla),$$

by Proposition 8. Thus

$$\nabla^* = -\text{div} : \boldsymbol{H}_0^1(\Omega) \to L^2(\Omega)$$

has range

$$\text{R}(-\text{div}) = \text{R}(\nabla^*) = \text{Ker }(\nabla)^\perp = L^2(\Omega)/\boldsymbol{R}.$$

□

We may choose an $\boldsymbol{u}_1 \in \boldsymbol{H}^1(\Omega)$ such that

$$\text{div } \boldsymbol{u}_0 - g = \text{div } \boldsymbol{u}_1.$$

Now if we set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_0 - \boldsymbol{u}_1$, then

$$\left.\begin{aligned} &-\nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} + \nu\Delta\left(\boldsymbol{u}_0 + \boldsymbol{u}_1\right) \in \boldsymbol{H}^{-1}(\Omega) \\ &\text{div } \boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{in } \Omega,$$
$$\boldsymbol{v} = 0 \qquad \text{on } \Gamma.$$

and the existence of $(\boldsymbol{v}, p)$ and thus $(\boldsymbol{u}, p)$ readily follows from Theorem 18. □

Remark 27. *If Ω is just bounded, open and Lipschitz, we may then take $\boldsymbol{\phi}$ as the trace of a $\boldsymbol{\phi}_0 \in \boldsymbol{H}^1(\Omega)$ such that*

$$\int_\Omega g dx = \int_\Omega div\ \boldsymbol{\phi}_0 dx,\ \boldsymbol{u} - \boldsymbol{\phi}_0 \in \boldsymbol{H}_0^1(\Omega),$$

and the conclusion still holds.

2.5. **Regularity results**

1. Elliptic regularity.

Lemma 28. *Suppose that* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *is a weak solution to the following Dirichlet problem:*

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \Gamma. \end{cases}$$

Then

$$\boldsymbol{f} \in \boldsymbol{H}^m(\Omega) \Rightarrow \boldsymbol{u} \in \boldsymbol{H}^{m+2}(\Omega).$$

2. L^p counterpart of the Stokes problem.

Proposition 29. *Let* Ω *be an open bounded set of class* C^r, $r = \max(m+2, 2)$, m *integer* > 0. *Let us suppose*

$$\boldsymbol{u} \in \boldsymbol{W}^{1,\alpha}(\Omega), \ p \in L^\alpha(\Omega), \ 1 < \alpha < \infty,$$

are solutions to (3). *If*

$$\boldsymbol{f} \in \boldsymbol{W}^{m,\alpha}(\Omega), \ g \in W^{m+1,\alpha}(\Omega), \ \boldsymbol{\phi} \in \boldsymbol{W}^{m+1-1/\alpha,\alpha}(\Gamma),$$

then

$$\boldsymbol{u} \in \boldsymbol{W}^{m+2,\alpha}(\Omega), \ p \in W^{m+1,\alpha}(\Omega), \tag{5}$$

and there exists a constant $c(\alpha, \nu, m, \Omega)$ *such that*

$$\begin{aligned} & \|\boldsymbol{u}\|_{\boldsymbol{W}^{m+2,\alpha}(\Omega)} + \|p\|_{W^{m+1,\alpha}(\Omega)/\boldsymbol{R}} \\ \leq \ & c_0 \left\{ \begin{array}{l} \|\boldsymbol{f}\|_{\boldsymbol{W}^{m,\alpha}(\Omega)} + \|g\|_{W^{m+1,\alpha}(\Omega)} \\ + \|\boldsymbol{\phi}\|_{\boldsymbol{W}^{m+1-1/\alpha,\alpha}(\Omega)} + d_\alpha \|\boldsymbol{u}\|_{\boldsymbol{L}^\alpha(\Omega)} \end{array} \right\}, \end{aligned} \tag{6}$$

where

$$d_\alpha = \begin{cases} 1, & \text{if } 1 < \alpha < 2, \\ 0, & \text{if } \alpha \geq 2. \end{cases}$$

Proof. This is an immediate consequence of Agmon-Douglish-Nirenberg. □

Remark 30. *For* $\alpha = 2$, $m \in \boldsymbol{R}$, $m \geq -1$, *one has results similar to those in Proposition 29 using interpolation techniques.*

3. A existence result.

Proposition 31. *Let* Ω *be an open set of class* C^r $r = \max(m+2, 2)$, m *integer* ≥ -1, *and*

$$\boldsymbol{f} \in \boldsymbol{W}^{m,\alpha}(\Omega),\ g \in W^{m+1,\alpha}(\Omega),\ \boldsymbol{\phi} \in \boldsymbol{W}^{m+2-1/\alpha,\alpha}(\Gamma),$$

be given satisfying the compatibility condition

$$\int_\Omega g dx = \int_\Gamma \boldsymbol{\phi} \cdot \boldsymbol{\nu} d\Gamma. \tag{7}$$

Then there exists a unique pair $(\boldsymbol{u}, p)$ $(p$ *is unique up to a constant*$)$ *which verifies* (3) *and satisfies* (5) *and* (6) *with* $d_\alpha = 0$ *for all* $1 < \alpha < \infty$.

2.6. Eigenfunctions of the Stokes problem

Let Ω be an open bounded domain in $\boldsymbol{R}^n$, and consider (1)

$$\left.\begin{array}{l} -\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \text{div } \boldsymbol{u} = 0 \end{array}\right\} \quad \text{in } \Omega,$$
$$\boldsymbol{u} = 0, \qquad \text{on } \Gamma.$$

By Theorem 18, we have the solution map

$$\boldsymbol{S} : L^2(\Omega) \ni \boldsymbol{f} \mapsto \boldsymbol{u} \in \boldsymbol{V}(\Omega) \subset\subset \boldsymbol{L}^2(\Omega)$$

is compact and injective. Also, $\boldsymbol{S}$ is self-adjoint in $\boldsymbol{L}^2(\Omega)$, due to direct computation as

$$\begin{aligned} (\boldsymbol{Sf}, \boldsymbol{g}) &= (\boldsymbol{u}, -\nu\boldsymbol{v} + \nabla q) \\ &= \nu\left(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}\right) \\ &= (-\nu\Delta\boldsymbol{u} + \nabla p, \boldsymbol{v}) \end{aligned}$$

$$= \quad (\boldsymbol{f}, \boldsymbol{S}\boldsymbol{g}),$$

if $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ are solutions of (1) with forces $\boldsymbol{f}$ and $\boldsymbol{g}$, respectively.

Thus by classical results in functional analysis,

$$\exists\, 0 < \mu_i \to 0,\ \{\boldsymbol{w}_i\} \text{ orthonormal in } \boldsymbol{L}^2(\Omega),\ s.t.\ \boldsymbol{w}_i \in \boldsymbol{V},\ \boldsymbol{S}\boldsymbol{w}_i = \mu_i \boldsymbol{w}_i,$$

i.e.

$$\lambda_i \boldsymbol{w}_i = S^{-1} \boldsymbol{w}_i,$$

with $0 < \lambda_i = 1/\mu_i \to \infty$.

Invoking Propositions 7 and 8, $\exists\, p_i \in \begin{cases} L^2_{loc}(\Omega), & \\ L^2(\Omega), & \text{if } \Omega \text{ is Lipschitz,} \end{cases}$ such that

$$\left.\begin{aligned} -\nu\Delta \boldsymbol{w}_i + \nabla p_i = \lambda_i \boldsymbol{w}_i \\ \text{div } \boldsymbol{w}_i = 0 \end{aligned}\right\} \quad \text{in } \Omega,$$
$$\boldsymbol{w}_i = 0, \qquad\qquad \text{on } \Gamma.$$

By Proposition 29, we then have

1.
$$\Omega \in C^m, m \geq 2 \Rightarrow \boldsymbol{w}_i \in \boldsymbol{H}^m(\Omega),\ p \in H^{m-1}(\Omega),$$

2.
$$\Omega \in C^\infty \Rightarrow \boldsymbol{w}_i \in C^\infty(\bar{\Omega}),\ p_i \in C^\infty(\bar{\Omega}).$$

Also, it is trivial that

$$(\boldsymbol{w}_i, \boldsymbol{w}_j) = \delta_{ij},\ (\nabla \boldsymbol{w}_i, \nabla \boldsymbol{w}_j) = \lambda_i \delta_{ij}.$$

3. **Slightly compressible fluids**

Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$. The stationary linearized equations of slightly compressible fluids are

$$\begin{cases} -\nu\Delta \boldsymbol{u}_\varepsilon - \frac{1}{\varepsilon}\nabla \text{div } \boldsymbol{u}_\varepsilon = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon = 0, & \text{on } \Gamma. \end{cases} \tag{1}$$

Here $\varepsilon > 0$ is small. (1) are also the stationary Lamé equations of elasticity.

By the projection Theorem 19, one easily verifies that for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$,

$$\exists\,|\ \boldsymbol{u}_\varepsilon \in \boldsymbol{H}_0^1(\Omega),\ s.t.\ (1) \text{ holds.}$$

Moreover, we have the following convergence result.

Theorem 32. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$, $\boldsymbol{u}_\varepsilon$, $\boldsymbol{u}$ are solutions of* (1), (1) *with the same $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, respectively. Then*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u},\ in\ \boldsymbol{H}_0^1(\Omega),$$

$$-\frac{div\ \boldsymbol{u}_\varepsilon}{\varepsilon} \to p\ in\ L^2(\Omega),$$

where p is the associated pressure to $\boldsymbol{u}$, and verifies

$$\int_\Omega p = 0.$$

Through the simple proof of Theorem 32, we use the following lemma, which has its own interest.

Lemma 33. *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^n$. Then*

$$\exists\ c(\Omega) > 0,\ s.t.\ \|p\|_{L^2(\Omega)} \leq c(\Omega)\left[\left|\int_\Omega p\right| + \|\nabla p\|_{\boldsymbol{H}^{-1}(\Omega)}\right]. \tag{2}$$

The proof of this lemma is omitted as a simple exercise in functional analysis.

Remark 34. *If Ω is not connected,* (2) *is true if we replace $\int_\Omega p$ by*

$$\sum_j \left|\int_{\Omega_j} p\right|$$

where Ω_j are the connected components of Ω. For extending Theorem 32 to this case, we just have to define p by imposing:

$$\int_{\Omega_j} p = 0,\ j.$$

FIGURE 1. J.L. Lions

4. **J.L. Lions**

Jacques-Louis Lions (3 May 1928-17 May 2001) was a French mathematician who made contributions to the theory of partial differential equations and to stochastic control, among other areas. He received the SIAM's John Von Neumann prize in 1986. Lions is listed as an ISI **highly cited researcher**.

After being part of the French Résistance in 1943 and 1944, J.-L. Lions entered the Ecole Normale Supérieure in 1947. Professor of mathematics at the Université of Nancy, the Faculty of Sciences of Paris, and the Ecole Polytechnique, he joined the prestigious Collége de France as well as the French Academy of Sciences in 1973. In 1979, he was appointed director of the Institut National de la Recherche en Informatique et Automatique (INRIA), where he taught and promoted the use of numerical simulations using finite elements integration. Throughout his career, Lions insisted on the use of mathematics in industry, with a particular involvement in the French space program, as well as in domains such as energy and the environment. This eventually led him to be appointed director of the Centre National d'Etudes Spatiales (CNES) from 1984 to 1992. Lions was elected President of the International Mathematical Union in 1991 and also received the Prize of Japan that same year. In 1992, the University of Houston awarded him an honorary doctoral degree. He was elected president of the French Academy of Sciences in 1996. He has left a considerable body of work, among this more than 400 scientific articles, 20 volumes of mathematics that were translated into English and Russian, and major contributions to several collective works, including the 4000 pages of the monumental Mathematical analysis and numerical methods for science and technology (in collaboration with Robert Dautray), as well as the Handbook of numerical analysis in 7 volumes (with Philippe G. Ciarlet).

His son Pierre-Louis Lions is also a well-known mathematician who was awarded a Fields Medal in 1994.

This follows from J.L. Lions.

Part 2. The steady-state Navier-Stokes equations

5. Existence and uniqueness theorems

5.1. Sobolev inequalities and compactness theorems

Lemma 35. *(Sobolev imbedding) For $u \in W^{m,p}(\boldsymbol{R}^n)$, we have*

1. *if $1/p - m/n = 1/q > 0$, then $u \in L^q(\boldsymbol{R}^n)$, and*
$$\left\|u\right\|_q \leq c(m,p,n)\left\|u\right\|_{m,p};$$
2. *if $1/p - m/n = 0$, then $u \in L^q(O)$, for any $1 \leq q < \infty$, $O \subset \boldsymbol{R}^n$ bounded, and*
$$\left\|u\right\|_{q,O} \leq c(m,p,n,q,O)\left\|u\right\|_{m,p};$$
3. *if $\boldsymbol{Z}/n \not\ni 1/p - m/n < 0$, then $u \in C^{k,\alpha}(\boldsymbol{R}^n)$ with*
$$k = [m - n/p]\,,\ \alpha = m - n/p - k,$$
and
$$\left\|u\right\|_{C^{k,\alpha}} \leq c(m,n,p)\left\|u\right\|_{m,p}.$$

Remark 36. 1. *For the case $1/p = m/n$, we have Orcliz imbeddings and BMO(VMO) imbeddings.*

2. *For the case $\boldsymbol{Z}/n \not\ni 1/p - 1/m < 0$, we have the Zygmund imbeddings.*
3. *This lemma deals with $\boldsymbol{R}^n$ case. For general open, smooth enough set $\Omega \subset \boldsymbol{R}^n$ with extension property, we have similar imbeddings.*
4. *If $u \in W_0^{m,p}(\Omega)$, then the imbeddings are valid without any hypothesis of Ω, such as smoothness, extension properties and the alike.*

5.2. **The homogeneous Navier-Stokes equations**

1. The problem.

   Let Ω be a Lipschitz, bounded open set in $\boldsymbol{R}^n$ with boundary Γ, and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. We consider the following homogeneous steady-state Navier-Stokes equations:

$$\left.\begin{array}{l} -\nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \text{div } \boldsymbol{u} = 0 \end{array}\right\} \text{ in } \Omega, \qquad \boldsymbol{u} = 0 \quad \text{on } \Gamma. \tag{1}$$

2. Weak formulation.

Definition 37. *A measurable vector $\boldsymbol{u}$ is said to be a weak solution to (1) if*

(a) $\boldsymbol{u} \in \boldsymbol{V}$;

(b) the following equality holds:

$$\nu\left(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}\right) + \left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right), \ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}; \tag{2}$$

Remark 38. *(a) By a simple density argument, (2) holds for all $\boldsymbol{v} \in \tilde{\boldsymbol{V}}$, which is the completion of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^n(\Omega)$. Observe that by Sobolev imbedding, $\tilde{\boldsymbol{V}} = \boldsymbol{V}$ for $n = 2, 3, 4$.*

(b) Once $\boldsymbol{u}$ is eatablished, the pressure $p \in L^1_{loc}(\Omega)$ is naturally associated by (2) and Proposition 7.

3. Properties of a trilinear map.

   Before going to the existence result, let us study the trilinear map:

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \left(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{w}\right),$$

appeared in (2).

One immediately verifies that

Lemma 39. *For any open set* $\Omega \subset \boldsymbol{R}^n$*,*

(a) b *is defined and trilinear continuous on*

$$\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \left(\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega)\right).$$

(b) b *is defined and trilinear continuous on* $\boldsymbol{V} \times \boldsymbol{V} \times \tilde{\boldsymbol{V}}$*.*

(c) if furthermore Ω *is bounded and* $n = 2$*, then* b *is defined and trilinear continuous on* $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$*.*

(d) if $n = 3, 4$*, then* b *is defined and trilinear continuous on* $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$*.*

(e) $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$*, for* $\boldsymbol{u} \in \boldsymbol{V}$*,* $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega)$*.*

(f) $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$*, for* $\boldsymbol{u} \in \boldsymbol{V}$*,* $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^n(\Omega)$*.*

4. Existence of a weak solution.

Theorem 40. *Let* Ω *be a bounded set in* $\boldsymbol{R}^n$*, and* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$*. Then the problem* (1) *has at least one weak solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times L^1_{loc}(\Omega)$*.*

Proof. We just invoke Galerkin method, and the solution of such an approximating algebraic system is ensured by the following

Lemma 41. *Let* X *be a finite dimensional Hilbert space with scalar product* $[\cdot, \cdot]$ *and norm* $[\cdot]$*, and* P *is a continuous mapping from* X *to itself such that*

$$[P(\xi), \xi] > 0, \text{ for } [\xi] = k > 0.$$

Then there exists a $\xi \in X$*,* $[\xi] \leq k$*, such that*

$$P(\xi) = 0.$$

Remark 42. *This is a high dimensional version of classical immediate value theorem for continuous functions, and may be proved by the Brouwer fixed point theorem.*

□

5. Restricted uniqueness.

Theorem 43. *If $n \leq 4$ and ν is sufficiently small or $\boldsymbol{f}$ is "sufficiently small so that*

$$\nu^2 > c(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{V}'}\,,$$

then there exists an unique solution $\boldsymbol{u}$ of (1).

Remark 44. *We need here $n \leq 4$ so that $\tilde{\boldsymbol{V}} = \boldsymbol{V}$, and hence we can choose the difference of two weak solutions as a test function in (2).*

5.3. The homogeneous Navier-Stokes equations (continued)

1. The unbounded case.

Let Ω be open unbounded in $\boldsymbol{R}^n$, we consider (1). As in Subsection 2.3, Chapter 1, we introduce

$$\boldsymbol{Y} = \text{the completion of } \boldsymbol{\mathcal{V}} \text{ under } \|\nabla\cdot\|_2\,,$$

and

$$\tilde{\boldsymbol{Y}} = \text{the completion of } \boldsymbol{\mathcal{V}} \text{ under } \|\nabla\cdot\|_2 + \|\cdot\|_n\,.$$

Recall that we have the continuous injection:

$$\boldsymbol{Y} \subset \left\{\boldsymbol{u} \in \boldsymbol{L}^{\frac{2n}{n-2}}(\Omega);\ \nabla\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\right\},$$

if $n \geq 3$.

Also, due to the fact Ω is unbounded, $\boldsymbol{Y}$ may not equal to $\tilde{\boldsymbol{Y}}$ even when $n \leq 4$. However, we have

Lemma 45. *For $n \geq 3$, the trilinear form b is continuous on $\boldsymbol{Y} \times \boldsymbol{Y} \times \tilde{\boldsymbol{Y}}$, and*

$$\begin{aligned} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) &= 0, && \boldsymbol{u} \in \boldsymbol{Y},\ \boldsymbol{v} \in \tilde{\boldsymbol{Y}}; \\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), && \boldsymbol{u} \in \boldsymbol{Y},\ \boldsymbol{v},\boldsymbol{w} \in \tilde{\boldsymbol{Y}}. \end{aligned}$$

Now, we state our existence result.

Theorem 46. *Let Ω be an open set in $\boldsymbol{R}^n$, $n \geq 3$, and $\boldsymbol{f} \in \boldsymbol{Y}'$. Then there exists at least one $\boldsymbol{u} \in \boldsymbol{Y}$ which verifies*

$$\nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \ \boldsymbol{v} \in \tilde{\boldsymbol{Y}}.$$

Proof. As in the proof of Theorem 40, we use Galerkin method. However, we should choose appropriate basis $\{\boldsymbol{w}_i\}$ in $\tilde{\boldsymbol{Y}}$, hence in $\boldsymbol{Y}$, for the sake of the well-definiteness of b.

Ultilizing Lemma 41, the approximation solutions $\{\boldsymbol{u}_n\} \subset \tilde{\boldsymbol{Y}}$ exist.

Now, the assumption that $\boldsymbol{f} \in \boldsymbol{Y}'$ is needed to do an a priori estimates for $\boldsymbol{u}_n$ in $\boldsymbol{Y}$ (clearly, it is hard to do such in $\tilde{\boldsymbol{Y}}$).

The passage to limit is then easy for test functions $\boldsymbol{V} \in \boldsymbol{\mathcal{V}}(\Omega)$, and a simple density conclude the proof. □

Remark 47. *(a) For $n = 2$, an element $\boldsymbol{u}$ of $\boldsymbol{Y}$ does not belong in general to any $L^\beta(\Omega)$ space. for this reason, the proof of Lemma 45 fails and b is not defined on $\boldsymbol{Y} \times \boldsymbol{Y} \times$ (some space).*

(b) As in the proof above, by a simple density argument, we may give

Definition 48. *Let Ω be open in $\boldsymbol{R}^n$, $n \geq 3$, and $\boldsymbol{f} \in \boldsymbol{Y}'$. Then a measurable vector $\boldsymbol{u}$ isa said to be a weak solution to (1) if*

(i) $\boldsymbol{u} \in \boldsymbol{Y}$;

(ii) the following equality holds:

$$\nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}.$$

2. Regularity of the solution.

Proposition 49. *Let Ω be open of class C^2 in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and $\boldsymbol{f} \in C^\infty(\bar{\Omega})$. Then any solution pair $(\boldsymbol{u}, p)$ of (1) belongs to $C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$.*

Proof. This is proved by bootstrap argument for bounded Ω, and cut-off function technique for the unbounded case. □

Remark 50. *(a) It is clear that we can assume less regularity of* $\boldsymbol{f}$ *and obtain less regularity of* $\boldsymbol{u}$ *and* p.

(b) the bootstrap argument fails when $n = 4$. *However, one can invoke a different method to show regularity.*

(c) If is an open question whether or not any weak solution to (1) *is regular, in case* $n \geq 5$.

5.4. **The non-homogeneous Navier-Stokes equations**

Let Ω be an open bounded domain in $\boldsymbol{R}^n$ of class C^2, and $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $\boldsymbol{\phi} = \text{rot}\,\boldsymbol{\xi}$, with

$$\boldsymbol{\xi} \in \boldsymbol{H}^2(\Omega),\ \nabla\boldsymbol{\xi} \in \boldsymbol{L}^n(\Omega),\ \boldsymbol{\xi} \in \boldsymbol{L}^\infty(\Omega). \tag{3}$$

Here rot denotes the usual rotational operator for $n = 2, 3$; for $n \geq 4$, rot denotes a linear differential operator with constant coefficients, such that div $(\text{rot}\,\boldsymbol{\xi}) = 0$.

We consider the following non-homogeneous steady-state Navier-Stokes problem:

$$\left.\begin{aligned} &-\nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \\ &\text{div}\ \boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{\phi}, \quad \text{on } \Gamma. \tag{4}$$

1. Existence result.

Theorem 51. *Under the hypotheses above, there exist at least one* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, *and a distribution* p *on* Ω, *such that* (4) *holds.*

Proof. The key ideas are:

(a) Using cut-off function technique, we can find a $\boldsymbol{\psi} \in \boldsymbol{V}(\Omega) \cap \boldsymbol{L}^n(\Omega)$ with $\boldsymbol{\psi} = \boldsymbol{\phi}$ on Γ, and

$$\|\rho\boldsymbol{\psi}\|_2 < \varepsilon,$$

for any sufficiently small $\varepsilon > 0$, where $\rho(\cdot) = \text{dist}\,(\cdot, \Gamma)$.

(b) Taking $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{\psi}$, (4) reduces to a homogenous equation, $(*)$ say.

(c) The resolution of $(*)$ is then solved by the projection theorem 19. The main ingredients are as follows:

$$\begin{aligned} |b(\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{u})| &= |b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})| \\ &\leq \|\nabla \boldsymbol{u}\|_2 \|\boldsymbol{u}\otimes\boldsymbol{\psi}\|_2 \\ &\leq \|\nabla \boldsymbol{u}\|_2 \left\|\frac{\boldsymbol{u}}{\rho}\right\|_2 \|\rho\boldsymbol{\psi}\|_2 \\ &\leq C\|\boldsymbol{u}\|_2^2 \|\rho\boldsymbol{\psi}\|_2 \\ &\leq C\varepsilon \|\nabla\boldsymbol{u}\|_2^2 . \end{aligned}$$

□

2. Regularity result.

Theorem 52. *Let $\Omega \in C^\infty$ in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and $\boldsymbol{f},\ \boldsymbol{\phi} \in C^\infty(\bar\Omega)$. Then the solution pair $(\boldsymbol{u},p)$ of (4) belongs to $C^\infty(\bar\Omega)\times C^\infty(\bar\Omega)$.*

3. Restrict uniqueness.

Theorem 53. *Suppose that $n\leq 4$, that $\|\boldsymbol{\phi}\|_n$ small so that*

$$|b(\boldsymbol{v},\boldsymbol{\phi},\boldsymbol{v})| \leq \frac{\nu}{2}\|\nabla\boldsymbol{v}\|_2^2,\ \boldsymbol{v}\in\boldsymbol{V},$$

and ν is sufficiently large so that

$$\nu^2 > 4c(n)\|\boldsymbol{f}+\nu\Delta\boldsymbol{\phi}-\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}\|_{\boldsymbol{V}'},$$

where $c(n)$ is the best constant in

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq c(n)\|\boldsymbol{u}\|_{\boldsymbol{H}_0^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)\cap\boldsymbol{L}^n(\Omega)}.$$

Then there exists an unique solution pair $(\boldsymbol{u},p)$ of (4).

6. Bifurcation theory and non-uniqueness results

We leave the interested reader to [4].

7. Roger Temam

Roger Meyer Temam (born 19 May 1940) is a College Professor of mathematics at The Indiana University, Bloomington. According to Mathematics Genealogy Project, (in the beginning of 2009) Temam has supervised 103 Ph.D. thesis; this is the highest number Ph.D. thesis supervised by an individual in the field of mathematics. He has a total of 281 mathematical descendants. He is known for his contribution to the theory of Navier-Stokes equations.

1. France.

   Temam was advised by Jacques-Louis Lions at the Université de Paris. He finished his dissertation in 1967. He was elected to the French Academy of Sciences on December 11, 2007 (while at Indiana University). From 1967 to 2003, Temam held a professorship at Universit Paris-Sud (Orsay).
2. Indiana.

   In the mid-1980s, Temam came to Indiana University to work with Ciprian Foias. Indiana made him a "a very nice offer." He taught at both Indiana and in France for some time. While at Indiana, Temam served as the director of the Institute for Scientific Computing and Applied Mathematics at IU.
3. Books.
   (a) R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, American Mathematical Society (2001).
   (b) R. Temam, Infinite Dimensonal Dynamical Systems in Mechanics and Physics, 2nd ed., Springer (1997).
   (c) C. Foias, O. Manley, R. Rosa, and R. Temam, Navier-Stokes Equations and Turbulence, Cambridge University Press (2001).
   (d) I. Ekeland and R. Témam, Convex Analysis and Variational Problems, Society for Industrial Mathematics (1987).
   (e) R. Temam and A. Miranville, Mathematical Modeling in Continuum Mechanics, 2nd ed., Cambridge University Press (2005).

(f) P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Springer-Verlag, Applies Mathematical Sciences Series, Vol.70 (1988).

4. Awards and honors.
   (a) Seymour Cray Prize in Numerical Simulation, 1989.
   (b) Elected to the French Academy of Sciences in December 2007.

Part 3. The evolutionary Navier-Stokes equations

8. The linear case

8.1. Some technical lemmas

Lemma 54. *Let X be a given Banach space with dual X' and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two functions belonging to $L^1(a,b;X)$. Then the following three conditions are equivalent:*

1. *$\boldsymbol{u}$ is a.e. equal to a primitive function of $\boldsymbol{v}$,*
$$\boldsymbol{u}(t)=\boldsymbol{\xi}+\int_a^t \boldsymbol{v}(s)ds,\ \boldsymbol{\xi}\in X,\ a.e.\ t\in[a,b]; \tag{1}$$
2. *for each test function $\phi\in\mathcal{D}((a,b))$,*
$$\int_a^b \boldsymbol{u}(t)\phi'(t)dt=-\int_a^b \boldsymbol{v}(t)\phi(t)dt\ \left(\phi'=\frac{d\phi}{dt}\right); \tag{2}$$
3. *for each $\eta\in X'$,*
$$\frac{d}{dt}\left\langle \eta,\boldsymbol{u}\right\rangle=\left\langle \eta,\boldsymbol{v}\right\rangle,\ in\ \mathcal{D}'((a,b)). \tag{3}$$

If one of the above items is satisfied, then $\boldsymbol{u}$, in particular, is equal to a continuous function from $[a,b]$ into X.

Proof. For simplicity, we assume $[a,b]=[0,T]$. And it is trivial that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

Hence we show

1. (3) $\Rightarrow$ (2).

Indeed,

$$\begin{aligned}\left\langle \eta, \int_0^T \boldsymbol{u}(t)\phi'(t)dt \right\rangle &= \int_0^T \langle \eta, \boldsymbol{u}(t)\rangle \phi'(t)dt \\ &= -\int_0^T \frac{d}{dt}\langle \eta, \boldsymbol{u}(t)\rangle \phi(t)dt \\ &= -\int_0^T \langle \eta, \boldsymbol{v}(t)\rangle \phi(t)dt \\ &= \left\langle \eta, -\int_0^T \boldsymbol{v}(t)\phi(t)dt \right\rangle,\ \eta \in X'.\end{aligned}$$

2. (2) $\Rightarrow$ (1).

Replacing $\boldsymbol{u}(t)$ by $\boldsymbol{u}(t) - \int_0^t \boldsymbol{v}(s)ds$, we need only prove that

$$\int_0^T \boldsymbol{v}(t)\phi'(t)dt = 0,\ \phi \in \mathcal{D}((0,T)) \Rightarrow \boldsymbol{v}(t) \text{ is a constant vector.}$$

While this follows from regularizaiton as

$$\begin{aligned} 0 &= \int_0^T \boldsymbol{v}(t)\left(\kappa_\varepsilon \star \phi\right)'(t)dt \ \ (\kappa \text{ an even mollifer}) \\ &= \int_0^T \boldsymbol{v}(t)\left(\kappa_\varepsilon \star \phi'\right)(t)dt \\ &= \int_0^T \left(\kappa_\varepsilon * \bar{\boldsymbol{v}}\right)(t)\phi'(t)dt \ \ (\bar{\boldsymbol{v}} \text{ the zero extension of } \boldsymbol{v}) \\ &= -\int_0^T \left(\kappa_\varepsilon \star \bar{\boldsymbol{v}}\right)'(t)\phi(t)dt,\ \phi \in \mathcal{D}((0,T)).\end{aligned}$$

□

Remark 55. *In the proof, we use the following observation:*

$$\left.\begin{array}{l} \boldsymbol{w} \in C([0,T];X) \\ \int_0^T \boldsymbol{w}(t)\phi(t)dt = 0,\ \phi \in \mathcal{D}((0,T)) \end{array}\right\} \Rightarrow \boldsymbol{w}(t) = 0,\ t \in [0,T].$$

Indeed, for $\eta \in X'$,

$$\begin{aligned} 0 &= \left\langle \eta, \int_0^T \boldsymbol{w}(t)\phi(t)dt \right\rangle \\ &= \int_0^T \langle \eta, \boldsymbol{w}(t)\rangle \phi(t)dt,\end{aligned}$$

which implies easily that

$$\langle \eta, \boldsymbol{w}(t)\rangle = 0,\ \eta \in X'.$$

The proof is then concluded by invoking Hahn-Banach Theorem. □

Lemma 56. *Let X, Y be two Banach spaces such that $X \subset Y$. Then*

$$\left.\begin{array}{l} \boldsymbol{u} \in L^\infty(0,T;X) \\ \boldsymbol{u} \in C([0,T];Y-w) \end{array}\right\} \Rightarrow \boldsymbol{u} \in C([0,T];X-w).$$

Proof. By replacing Y by the closure of X in Y, we may suppose that X is dense in Y, and hence

$$Y' \subset X'$$

is a dense continuous injection.

For any $\eta \in X'$, any $\varepsilon > 0$,

$$\exists\, \eta_\varepsilon \in Y',\ s.t.\ \|\eta_\varepsilon - \eta\|_{X'} < \frac{\varepsilon}{2\,\|\boldsymbol{u}\|_{L^\infty(0,T;X)}},$$

thus

$$\begin{aligned} |\langle \eta, \boldsymbol{u}(t) - \boldsymbol{u}(t_0)\rangle| &\leq |\langle \eta - \eta_\varepsilon, \boldsymbol{u}(t) - \boldsymbol{u}(t_0)\rangle| + |\langle \eta_\varepsilon, \boldsymbol{u}(t) - \boldsymbol{u}(t_0)\rangle| \\ &\leq 2\,\|\boldsymbol{u}\|_{L^\infty(0,T;X)}\,\|\eta_\varepsilon - \eta\|_{X'} + \frac{\varepsilon}{2}\ \ (|t-t_0| \text{ small}) \\ &< \varepsilon. \end{aligned}$$

The proof is concluded. □

Lemma 57. *Let X, Y be two Hilbert spaces satisfying*

$$X \subset Y \equiv Y' \subset X',$$

with each continuous inclusion dense. If a function $\boldsymbol{u} \in L^2(0,T;X) \cap L^\infty(0,T;Y)$ and $\boldsymbol{u}' \in L^\infty(0,T;X')$, then $\boldsymbol{u} \in C([0,T];Y)$ with

$$\frac{d}{dt}\|\boldsymbol{u}\|_Y^2 = 2\,\langle \boldsymbol{u}', \boldsymbol{u}\rangle_{X'\times X}\,,\ in\ \mathcal{D}'((0,T)). \tag{4}$$

Proof. (4) is prove easily be regularization. Also, by Lemma 56, $\boldsymbol{u} \in C([0,T]; X' - w)$, and hence

$$
\begin{aligned}
& \|\boldsymbol{u}(t) - \boldsymbol{u}(t_0)\|_Y^2 \\
= \; & \|\boldsymbol{u}(t)\|_Y^2 + \|\boldsymbol{u}(t_0)\|_Y^2 - 2\left(\boldsymbol{u}(t), \boldsymbol{u}(t_0)\right)_Y \\
= \; & \left[\|\boldsymbol{u}(t_0)\|_Y^2 + 2\int_{t_0}^{t} \langle \boldsymbol{u}'(s), \boldsymbol{u}(s)\rangle \, ds\right] + \|\boldsymbol{u}(t_0)\|_Y^2 - 2\left(\boldsymbol{u}(t), \boldsymbol{u}(t_0)\right)_Y \\
= \; & 2\left[\|\boldsymbol{u}(t_0)\|_Y^2 - \langle \boldsymbol{u}(t), \boldsymbol{u}(t_0)\rangle_{X' \times X}\right] + 2\int_{t_0}^{t} \langle \boldsymbol{u}'(s), \boldsymbol{u}(s)\rangle \, ds \\
\to \; & 0, \quad \text{as } t \to t_0,
\end{aligned}
$$

which verifies that $\boldsymbol{u} \in C([0,T]; Y)$. □

8.2. The existence and uniqueness theorem

1. The problem.

   Let Ω be a Lipschitz open bounded set in $\boldsymbol{R}^n$, $T > 0$. We consider the following linearized evolutionary Navier-Stokes equations:

$$
\left.
\begin{aligned}
\partial_t u - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \\
\text{div } \boldsymbol{u} = 0
\end{aligned}
\right\} \quad \text{in } Q \equiv \Omega \times (0,T),
$$
$$
\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega \times [0,T],
$$
$$
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad \text{in } \Omega. \tag{5}
$$

   Here $\boldsymbol{u}$ is the velocity filed, $\nu > 0$ is the kinematic viscosity, p is a scalar pressure, $\boldsymbol{f}$ is the external force, and $\boldsymbol{u}_0$ is the initial velocity field.

2. Weak formulation.

   Definition 58. *Given $\boldsymbol{f} \in L^2(0,T; \boldsymbol{V}')$, $\boldsymbol{u}_0 \in \boldsymbol{H}$, a measurable function $\boldsymbol{u}$ is said to be a weak solution of (5) if*

   (a) $\boldsymbol{u} \in C([0,T]; \boldsymbol{H}) \cap L^2(0,T; \boldsymbol{V})$;

   (b)

$$
\begin{cases}
\frac{d}{dt}\left(\boldsymbol{u}, \boldsymbol{v}\right) + \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) = \left\langle \boldsymbol{f}, \boldsymbol{v}\right\rangle, \boldsymbol{v} \in \boldsymbol{V}, \ \text{in } \mathcal{D}'((0,T)) \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \ a.e.
\end{cases} \tag{6}
$$

3. Existence of an unique weak solution.

Theorem 59. *For $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$, and $\boldsymbol{u}_0 \in \boldsymbol{H}$, there exists an unique weak solution $\boldsymbol{u}$ on $[0,T]$ of (5).*

Proof. (a) Existence.

We use Faedo-Galerkin method.Choose a basis $\{\boldsymbol{w}_i\} \subset \boldsymbol{V}$, and define approximate problem as follows:

$$\begin{cases} \boldsymbol{u}_m = g_m^j(t)\boldsymbol{w}_j, \\ (\boldsymbol{u}_m', \boldsymbol{w}_j) + \nu\,(\nabla \boldsymbol{u}_m, \boldsymbol{w}_j) = \langle \boldsymbol{f}_m, \boldsymbol{w}_j \rangle\,, \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \end{cases} \tag{7}$$

where $\boldsymbol{f}_m$, $\boldsymbol{u}_{0m}$ are smooth functions defined in $[0,T]$ with values in $\boldsymbol{V}'$, $\boldsymbol{H}$ respectively, and converge to $\boldsymbol{f}$, $\boldsymbol{u}_0$ in their own function classes.

The resolution of (7) is easily solved, and we have the following a priori uniform estimates:

$$\boldsymbol{u}_m \text{ uniformly bounded in } L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}).$$

Passage to limit gives an

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$$

satisfying (6).

Up to now, what left to check is

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}).$$

While this follows from the fact

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),\ \boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}')\ \text{(by (6)}_1\text{)}\,,$$

and Lemma 57.

(b) Uniqueness.

Observe that $(6)_1$ can be rewrited as

$$\boldsymbol{u}' + \nu \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f},$$

with $\boldsymbol{A} : \boldsymbol{V} \to \boldsymbol{V}'$ being linear and continuous. The uniqueness then follows from standard trick.

□

4. Miscellaneous remarks.
   (a) More general foces.
   If we assume in general that
$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}) + L^1(0,T;\boldsymbol{H}),$$
   then Theorem 59 holds with
$$\boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}) + L^1(0,T;\boldsymbol{H}).$$
   (b) The case Ω is unbounded.
   If Ω is unbounded, we need no longer introduce $\boldsymbol{Y}$, which is the completion of $\boldsymbol{\mathcal{V}}$ under $\|\nabla\cdot\|_2$, due to a simple Young inequality. And Theorem 59 still holds.
   (c) Interpretation of the weak solution.
      (i) The pressure p is naturally associated to $\boldsymbol{u}$, due to Proposition 7.
      (ii) By Proposition 14,
$$\boldsymbol{V} = \left\{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega);\ \operatorname{div} \boldsymbol{u} = 0\right\},$$
      we have $\boldsymbol{u} = 0$ on $\partial\Omega \times [0,T]$ is a trace sense and div $\boldsymbol{u} = 0$ in Q, in a distributional sense.
      (iii) From the fact
$$\boldsymbol{u} \in C([0,T];\boldsymbol{L}^2(\Omega)),$$
      we gather that $\boldsymbol{u}(t)$ tends to $\boldsymbol{u}_0$ in L^2-norm, as $t \to 0_+$.
   (d) Regularity of the unique weak solution.

Proposition 60. *Let us assume that Ω is of class C^2, that*
$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}),\ \boldsymbol{u}_0 \in \boldsymbol{V}.$$

Then

$$\begin{cases} \boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^2(\Omega)), \\ \boldsymbol{u}' \in L^2(0,T;\boldsymbol{H}) \ \ (i.e.\ \boldsymbol{u}' \in \boldsymbol{L}^2(Q)) \\ p \in Ł^2(0,T;\boldsymbol{H}^1(\Omega)). \end{cases} \tag{8}$$

Proof. (i) $(8)_2$ holds.

This follows from a priori estimates of $\boldsymbol{u}_m'$ in the Faedo-Galerkin approximate problem.

(ii) $(8)_{1,3}$ hold.

Indeed, this follows from classical regularity theorem for the steady-state Stokes problem, Proposition 29 and the established $(8)_2$.

□

9. Compactness theorems

9.1. An ε-type inequality

Lemma 61. *Let X_0, X, X_1 be three Banach spaces such that*

$$X_0 \subset\subset X \subset X_1.$$

Then

$$\forall\, \varepsilon > 0,\ \exists\, C_\varepsilon > 0,\ s.t.\ \|\boldsymbol{u}\|_X \leq \varepsilon \|\boldsymbol{u}\|_{X_0} + C_\varepsilon \|\boldsymbol{u}\|_{X_1},\ \boldsymbol{u} \in X_0.$$

Proof. This is proved by contradiction. □

9.2. A compactness theorem in Banach spaces

Theorem 62. *Let*

- X_0, X, X_1 *be three Banach spaces with* X_0, X_1 *reflexive, and*

$$X_0 \subset\subset X \subset X_1;$$

- $0 < T < \infty$, $1 < \alpha_1, \alpha_2 < \infty$.

Then

$$\begin{aligned}\boldsymbol{\mathcal{Y}} &\equiv \boldsymbol{\mathcal{Y}}(0,T;\alpha_0,\alpha_1;X_0,X_1)\\ &\equiv \left\{\boldsymbol{u}\in L^{\alpha_0}(0,T;X_0);\boldsymbol{u}'=\frac{d\boldsymbol{u}}{dt}\in L^{\alpha_1}(0,T;X_1)\right\}\\ &\subset\subset L^{\alpha_0}(0,T;X).\end{aligned}$$

Proof. 1. Let $\{\boldsymbol{u}_m\}\subset\boldsymbol{\mathcal{Y}}$ be bounded, then $(1<\alpha_0,\alpha_1<\infty, X_1, X_2$ reflexive)

$$\begin{aligned}\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u} \text{ in } L^{\alpha_0}(0,T;X_0),\\ \boldsymbol{u}_m' &\rightharpoonup \boldsymbol{u}' \text{ in } L^{\alpha_1}(0,T;X_1),\end{aligned}$$

up to some subsequence. Denote by

$$\boldsymbol{v}_m=\boldsymbol{u}_m-\boldsymbol{u},$$

then we need only show

$$\boldsymbol{v}_m\to 0 \text{ in } L^{\alpha_0}(0,T;X). \tag{1}$$

2. Due to Lemma 61, (1) reduces to

$$\boldsymbol{v}_m\to 0 \text{ in } L^{\alpha_0}(0,T;X_1). \tag{2}$$

Due also to the fact

$$\boldsymbol{\mathcal{Y}}\subset C([0,T];X_1),$$

and thus

$$\|\boldsymbol{v}_m\|_{X_1}\le C<\infty, \tag{3}$$

(2) reduces, even further, by Lebesgue's dominated convergence theorem, to

$$\|\boldsymbol{v}_m\|_{X_1}\to 0, \text{ a.e.} \tag{4}$$

3. We may just prove (4) in case $t=0$, with other $t\in(0,T]$ easily modified. Observing that

$$\boldsymbol{v}_m(0)=\boldsymbol{v}_m(t)-\int_0^t \boldsymbol{v}_m'(\tau)d\tau,$$

$$\begin{aligned}\boldsymbol{v}_m(0) &= \frac{1}{s}\int_0^s \boldsymbol{v}_m(t)dt - \frac{1}{s}\int_0^s\int_0^t \boldsymbol{v}_m'(\tau)d\tau dt\\ &= \frac{1}{s}\int_0^s \boldsymbol{v}_m(t)dt - \frac{1}{s}\int_0^s (s-\tau)\,\boldsymbol{v}_m'(\tau)d\tau\\ &\equiv \boldsymbol{a}_m + \boldsymbol{b}_m,\end{aligned}$$

we have for $\forall\, \varepsilon > 0$, $\exists\, s > 0$ small, such that

$$\|\boldsymbol{v}_m\|_{X_1} \leq \int_0^s \|\boldsymbol{v}_m'(\tau)\|_{X_1}\, d\tau \leq s^{1-1/\alpha_1}\, \|\boldsymbol{v}_m\|_{L^{\alpha_1}(0,T;X_1)} < \varepsilon/2.$$

For this $s > 0$, we choose m large enough so that

$$\begin{aligned}& \boldsymbol{v}_m(t) \rightharpoonup 0 \text{ in } X_0, \text{ a.e. } t\in[0,s]\\ \Rightarrow\ & \boldsymbol{v}_m(t) \to 0 \text{ in } X, \text{ a.e. } t\in[0,s]\ \ (X_0 \subset\subset X_1)\\ \Rightarrow\ & \|\boldsymbol{a}_m\|_{X_1} < \varepsilon/2\end{aligned}$$

(by Lebesgue's dominated convergence theorem and (3)).

This concludes the proof.

□

9.3. A compactness theorem involving fractional derivatives

1. Fourier transform and fractional derivative in Hilbert spaces. Let H be a Hilbert space, and $\boldsymbol{v} : \boldsymbol{R} \to H$ be a function, if

$$\hat{\boldsymbol{v}}(\tau) \equiv \int_{-\infty}^{\infty} e^{-2\pi i\tau t}\boldsymbol{v}(t)dt$$

is defined $a.e.\ \tau \in R$, then $\hat{\boldsymbol{v}}$ is said to be the Fourier transform of $\boldsymbol{v}$.

With this definition, we may define the derivative of $\boldsymbol{v}$ of order γ as

$$\widehat{D^\gamma \boldsymbol{v}}(\tau) = (2\pi i\tau)^\gamma\, \hat{\boldsymbol{v}}(\tau).$$

2. A compactness theorem via fractional derivative.

Theorem 63. *Let X_0, X, X_1 be three Hilbert spaces satisfying*

$$X_0 \subset\subset X \subset X_1.$$

Then for any bounded $K \subset \boldsymbol{R}$, *and* $\gamma > 0$,

$$H_K^\gamma(\boldsymbol{R}; X_0, X_1) \subset\subset L^2(\boldsymbol{R}; X).$$

Here

$$H_K^\gamma(\boldsymbol{R}; X_0, X_1) \equiv \{\boldsymbol{u} \in H^\gamma(\boldsymbol{R}; X_0, X_1);\ \textit{supp}\ \boldsymbol{u} \subset K\},$$

$$H^\gamma(\boldsymbol{R}; X_0, X_1) \equiv \left\{\boldsymbol{u} \in L^2(\boldsymbol{R}; X_0);\ D^\gamma \boldsymbol{u} \in L^2(\boldsymbol{R}; X_1)\right\}.$$

Proof. (a) Let $\{\boldsymbol{u}_m\} \subset H_K^\gamma(\boldsymbol{R}; X_0, X_1)$ be bounded, then

$$\begin{aligned} &\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ in } L^2(\boldsymbol{R}; X_0), \\ &D^\gamma \boldsymbol{u}_m \rightharpoonup D^\gamma \boldsymbol{u} \text{ in } L^2(\boldsymbol{R}; X_1). \end{aligned}$$

Denote by

$$\boldsymbol{v}_m = \boldsymbol{u}_m - \boldsymbol{u},$$

we have

$$\begin{aligned} &\boldsymbol{v}_m \rightharpoonup 0 \text{ in } L^2(\boldsymbol{R}; X_0); \\ &|\cdot|^\gamma\, \hat{\boldsymbol{v}}_m \rightharpoonup 0 \text{ in } L^2(\boldsymbol{R}; X_1). \end{aligned} \tag{5}$$

We need only show that

$$\boldsymbol{v}_m \to 0 \text{ in } L^2(\boldsymbol{R}; X). \tag{6}$$

(b) As in the proof of Theorem 62, we invoke Lemma 61 to reduce (6) to

$$\boldsymbol{v}_m \to 0 \text{ in } L^2(\boldsymbol{R}; X_1). \tag{7}$$

(c) In order to prove (7), we calculate as

$$\begin{aligned} \|\boldsymbol{v}_m\|^2_{L^2(\boldsymbol{R};X_1)} &= \|\hat{\boldsymbol{v}}_m\|^2_{L^2(\boldsymbol{R};X_1)} \text{ (Parseval identity)} \\ &= \int_{-\infty}^{\infty} \|\hat{\boldsymbol{v}}(\tau)\|^2_{X_1}\, d\tau \\ &= \int_{|\tau|\le M} \|\hat{\boldsymbol{v}}(\tau)\|^2_{X_1}\, d\tau + \int_{|\tau|>M} \frac{\left(1+|\tau|^{2\gamma}\right)\|\hat{\boldsymbol{v}}(\tau)\|^2_{X_1}}{1+|\tau|^{2\gamma}} d\tau \\ &\le \int_{|\tau|\le M} \|\hat{\boldsymbol{v}}(\tau)\|^2_{X_1}\, d\tau + \frac{\|\boldsymbol{v}\|^2_{H^\gamma(\boldsymbol{R};X_0,X_1)}}{1+M^{2\gamma}} \end{aligned}$$

$$\equiv \quad I_1 + I_2.$$

For any $\varepsilon > 0$, we may choose $M > 0$ large, so that

$$I_2 < \varepsilon/2,$$

and thus we need only show for this M, for large m,

$$I_1 < \varepsilon/2. \tag{8}$$

(d) As in the proof of Theorem 62, we use Lebesgue's dominated convergence theorem and the fact $X_0 \subset X_1$ to show (8), thus conclude the proof of Theorem 63.

In fact, the following items leads to (8) as depicted.

(i) $\hat{\boldsymbol{v}}_m(\tau) \rightharpoonup 0$ in X_0 a.e. τ.

For any $\boldsymbol{w} \in X_0$,

$$\langle \hat{\boldsymbol{v}}(\tau), \boldsymbol{w} \rangle_{X_0} = \int_{-\infty}^{\infty} e^{-2\pi i \tau t} \langle \boldsymbol{v}_m(t), \boldsymbol{w} \rangle_{X_0} \, dt \to 0,$$

by $(5)_1$.

(ii) By compact imbedding

$$X_0 \subset\subset X_1,$$

we have

$$\hat{\boldsymbol{v}}_m(\tau) \to 0 \text{ in } X_1, \ a.e.$$

(iii) The boundedness of $\hat{\boldsymbol{v}}$ in X_1.

$$\begin{aligned}
\|\hat{\boldsymbol{v}}(\tau)\|_{X_1} &= \left\| \int_{-\infty}^{\infty} e^{-2\pi i \tau t} \chi_K(t) \boldsymbol{v}(t) dt \right\|_{X_1} \\
&\leq \int_{-\infty}^{\infty} \chi_K(t) \|\boldsymbol{v}(t)\|_{X_1} dt \\
&\leq \|\chi_K\|_{L^2(\boldsymbol{R})} \|\boldsymbol{v}\|_{L^2(\boldsymbol{R};X)} \\
&< \infty.
\end{aligned}$$

□

More generally, we have the following

Theorem 64. *Let X_0, X be two Banach spaces, X_1 be Hilbert spaces satisfying*

$$X_0 \subset\subset X \subset X_1.$$

Then

$$\begin{aligned} & \boldsymbol{\mathcal{Y}}_K^\gamma\left(\boldsymbol{R};\alpha_0,2;X_0,X_1\right)\ (1<\alpha_0<\infty) \\ \equiv\ & \left\{\boldsymbol{u}\in L^{\alpha_0}(\boldsymbol{R};;X_0);\ D^\gamma\boldsymbol{u}\in L^2(0,T;X_1),\ \text{supp}\ \boldsymbol{u}\subset K\right\} \\ \subset\subset\ & L^{\alpha_0}(\boldsymbol{R};X). \end{aligned} \tag{9}$$

3. A critical case of compact theorem 62.

Theorem 65. *Let X_0, X be two Banach spaces, X_1 be a Hilbert space, and*

$$X_0 \subset\subset X \subset X_1.$$

Then

$$\begin{aligned} \boldsymbol{\mathcal{Y}} &\equiv \boldsymbol{\mathcal{Y}}\left(0,T;\alpha_0,1;X_0,X_1\right)\ (1<\alpha_0<\infty) \\ &\equiv \left\{\boldsymbol{u}\in L^{\alpha_0}(0,T;X_0);\ \boldsymbol{u}'=\frac{d\boldsymbol{u}}{dt}\in L^1(0,T;X_1)\right\} \\ &\subset\subset L^{\alpha_0}(0,T;X). \end{aligned}$$

Proof. 1. We shall show that

$$\boldsymbol{\mathcal{Y}}\subset\boldsymbol{\mathcal{Y}}^\gamma_{[0,T]}\left(\boldsymbol{R};\alpha_0,2;X_0,X_1\right), \tag{10}$$

where, we recall, the latter space is defined in (9). And Theorem 65 follows readily from Theorem 64.

2. Notations.

For a function $\boldsymbol{v}:[0,T]\to Y$, where Y is a Banach space, we denote its zero extension to $\boldsymbol{R}$ by $\tilde{v}$. Obviously,

$$\tilde{\boldsymbol{v}}'(t)=\tilde{\boldsymbol{v}}'(t)+\boldsymbol{v}(0)\delta_0-\boldsymbol{v}(T)\delta_T. \tag{11}$$

3. We begin to show (10). Let $\boldsymbol{u} \in \boldsymbol{\mathcal{Y}}$, then

$$\boldsymbol{u} \in L^{\alpha_0}(0,T;X_0),\ \boldsymbol{u}' \in L^1(0,T;X_1),$$

and thus

$$\tilde{\boldsymbol{u}} \in L^{\alpha_0}(\boldsymbol{R};X_0),\ \tilde{\boldsymbol{u}'} \in L^1(\boldsymbol{R};X_1). \tag{12}$$

From (11),

$$\tilde{\boldsymbol{u}}' = \tilde{\boldsymbol{u}'} + \boldsymbol{u}(0)\delta_0 - \boldsymbol{u}(T)\delta_T. \tag{13}$$

Applying Fourier transform on both sides of (13), we deduce

$$\begin{aligned} 2\pi i\tau\hat{\tilde{\boldsymbol{u}}}(\tau) &= \widehat{\tilde{\boldsymbol{u}'}}(\tau) + \boldsymbol{u}(0) - \boldsymbol{u}(T)e^{-2\pi i\tau T} \\ &\in L^\infty(0,T;X_1)\ \text{(by (12))}\,. \end{aligned} \tag{14}$$

4. We then show that for some $\gamma > 0$ small,

$$D^\gamma\tilde{\boldsymbol{u}} \in L^2(\boldsymbol{R};X_1).$$

In fact,

$$\begin{aligned} \|D^\gamma\tilde{\boldsymbol{u}}\|^2_{L^2(\boldsymbol{R};X_1)} &= \int_{-\infty}^{\infty} |\tau|^{2\gamma} \left\|\hat{\tilde{\boldsymbol{u}}}\right\|^2_{X_1} d\tau \\ &\leq c(\gamma)\int_{-\infty}^{\infty} \frac{1+|\tau|^2}{1+|\tau|^{2(1-\gamma)}} \left\|\hat{\tilde{\boldsymbol{u}}}\right\|^2_{X_1} d\tau\ \ \text{(by Young inequality)} \\ &\leq c(\gamma)\int_{-\infty}^{\infty} \left\|\hat{\tilde{\boldsymbol{u}}}\right\|^2_{X_1} d\tau \\ &\quad + c(\gamma)\int_{-\infty}^{\infty} \frac{1}{1+|\tau|^{2(1-\gamma)}} d\tau \cdot \sup_{\tau\in\boldsymbol{R}} \left[|\tau| \left\|\hat{\tilde{\boldsymbol{u}}}(\tau)\right\|_{X_1}\right] \\ &< \infty,\ \text{(by (14))} \end{aligned}$$

if

$$\gamma < 1/2.$$

□

10. **Existence and uniqueness theorems** $(n \leq 4)$

10.1. **An existence theorem in** $\mathbb{R}^n (n \leq 4)$

1. The problem.

   Let Ω be an open Lipschitz, bounded set, we consider

$$\left.\begin{aligned} &\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } Q = \Omega \times (0,T), \qquad (1)$$
$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega \times (0,T),$$
$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad \text{in } \Omega.$$

   Here $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are given, defined on $\Omega \times [0,T]$ and Ω, respectively.

2. Weak formulation.

   Definition 66. *Let $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$, $\boldsymbol{u}_0 \in \boldsymbol{H}$, a measurable vector $\mathbf{u}$ defined on $\Omega \times [0,T]$ is said to be a weak solution of* (1) *if*

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}_w) \cap L^2(0,T;\boldsymbol{V}), \qquad (2)$$

$$\boldsymbol{u}' + \nu \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}, \text{ on } (0,T), \qquad (3)$$

$$\boldsymbol{u}(0) = u_0. \qquad (4)$$

   Here

$$\begin{aligned} \boldsymbol{A} : \ &\boldsymbol{V} \ \to \ \boldsymbol{V}' \\ &\boldsymbol{u} \ \mapsto \ [\boldsymbol{V} \ni \boldsymbol{v} \mapsto (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \in \boldsymbol{R}], \end{aligned}$$

$$\begin{aligned} \boldsymbol{B} : \ &\boldsymbol{V} \ \to \ \boldsymbol{V}' \\ &\boldsymbol{u} \ \mapsto \ [\boldsymbol{V} \ni \boldsymbol{v} \mapsto b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{R}]. \end{aligned}$$

   Remark 67. *We may consider more general force*

$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}') + L^1(0,T;\boldsymbol{H}).$$

3. The existence result.

Theorem 68. *There exists at least one weak solution to* (1) *on* $[0, T]$.

The proof of Theorem 68 will be developed in Subsection 10.2.

10.2. **Proof of Theorem 68**

We apply the Galerkin procedure. This key points are listed as follows.

1. Construct approximates solutions $\{\boldsymbol{u}_m\}$. The local existence involves ODE theory.
2. Establish the following a priori estimate:
$$\boldsymbol{u}_m \text{ uniformly bounded in } L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T, \boldsymbol{V}), \tag{5}$$
which yields that $\boldsymbol{u}_m$ is globally defined (on $[0, T]$).
3. Applying Fourier transform, tracking the proof of Theorem 63, we can show that
$$\boldsymbol{u}_m \text{ uniformly bounded in } H^\gamma_{[0,T]}(\boldsymbol{R}; \boldsymbol{V}, \boldsymbol{H}). \tag{6}$$
Then applying Theorem 63, we have
$$\boldsymbol{u}_m \to \boldsymbol{u} \text{ in } L^2(0, T; \boldsymbol{H}), \tag{7}$$
for some
$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V}). \tag{8}$$
4. Obviously, (4) is true. With (7), we easily show (3). Meanwhile, (2) follows from (8), (3), and Lemma 56.

10.3. **Regularity and uniqueness ($n = 2$)**

1. Some inequalities.

Lemma 69. *If* $n = 2$, *for any open set* Ω,
$$\|v\|_{L^4(\Omega)} \leq 2^{1/4} \|v\|^{1/2}_{L^2(\Omega)} \|\nabla v\|^{1/2}_{L^2(\Omega)}, \ v \in H^1_0(\Omega). \tag{9}$$

Proof. It suffices to prove (9) for $v \in C_c^1(\Omega)$. For such a v, we write

$$v^2(x) = 2\int_{-\infty}^{x_1} v\partial_1 v dx_1 \le 2\int_{\boldsymbol{R}} |v\partial_1 v|\, dx_1 \equiv v_1(x_2),$$

$$v^2(x) = 2\int_{-\infty}^{x_2} v\partial_2 v dx_2 \le 2\int_{\boldsymbol{R}} |v\partial_2 v|\, dx_2 \equiv v_2(x_1),$$

and thus

$$\begin{aligned}
\int_\Omega |v|^4\, dx &= \int_{\boldsymbol{R}^2} |v|^4\, dx \\
&\le \int_{\boldsymbol{R}^2} v_1(x_2)v_2(x_1)dx \\
&\le \int_{\boldsymbol{R}} v_1(x_2)dx_2 \cdot \int_{\boldsymbol{R}} v_2(x_1)dx_1 \\
&\le 4\int_{\boldsymbol{R}^2} |v\partial_1 v|\, dx \cdot \int_{\boldsymbol{R}^2} |v\partial_2 v|\, dx \\
&\le 4\|v\|_{L^2(\boldsymbol{R}^2)}^2 \|\partial_1 v\|_{L^2(\boldsymbol{R}^2)} \|\partial_2 v\|_{L^2(\boldsymbol{R}^2)} \\
&\le 2\|v\|_{L^2(\boldsymbol{R}^2)}^2 \|\nabla v\|_{L^2(\boldsymbol{R}^2)}^2 \\
&= 2\|v\|_{L^2(\Omega)}^2 \|\nabla v\|_{L^2(\Omega)}^2 .
\end{aligned}$$

□

Lemma 70. *If $n = 2$,*

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le 2^{1/2} \|\boldsymbol{u}\|_2^{1/2} \|\nabla\boldsymbol{u}\|_2^{1/2} \|\nabla\boldsymbol{v}\|_2^{1/2} \|\boldsymbol{w}\|_2^{1/2} \|\nabla\boldsymbol{w}\|_2^{1/2} . \tag{10}$$

If $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$, then $\boldsymbol{B}\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}')$, and

$$\|\boldsymbol{B}\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{V}')} \le 2^{1/2} \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H})} \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{V})} . \tag{11}$$

Proof. Direct computation using Hölder inequality and Lemma 69 shows that

$$\begin{aligned}
|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le \|\boldsymbol{u}\|_4 \|\nabla\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_4 \\
&\le 2^{1/2} \|\boldsymbol{u}\|_2^{1/2} \|\nabla\boldsymbol{u}\|_2^{1/2} \|\nabla\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_2^{1/2} \|\nabla\boldsymbol{w}\|_2^{1/2} .
\end{aligned}$$

We now proceed to prove (11). In fact, for general n, we have

$$\langle \boldsymbol{B}\boldsymbol{u},\boldsymbol{v}\rangle = b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = -b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{u}) \le \|\boldsymbol{u}\|_4^2 \|\boldsymbol{v}\|_2$$

$$\Rightarrow \begin{cases} \|\boldsymbol{Bu}\|_{\boldsymbol{V}'} \leq C(n)\|\boldsymbol{u}\|_4^2 \\ \leq C(n)\|\boldsymbol{u}\|_2^{2-n/2}\|\nabla\boldsymbol{u}\|_2^{n/2} \in L^{4/n}(0,T). \end{cases} \tag{12}$$

And in case $n=2$, $C(n)=2^{1/2}$, $4/n=2$. □

2. Regularity and uniqueness result.

Theorem 71. *If $n=2$, then the weak solution $\boldsymbol{u}$ to (1) given by Theorem 68 is unique. Moreover, $\boldsymbol{u}\in C([0,T];\boldsymbol{H})$, and $\lim_{t\to 0}\boldsymbol{u}(t)=\boldsymbol{u}_0$, in $\boldsymbol{H}$.*

Proof. (a) $\boldsymbol{u}\in C([0,T],\boldsymbol{H})$.

In fact,

$$\left.\begin{array}{c} \boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V}) \\ \boldsymbol{u}_t=\boldsymbol{f}-\nu\boldsymbol{Au}-\boldsymbol{Bu}\in L^2(0,T;\boldsymbol{V}')\ \text{(by (11))} \\ \text{Lemma 57} \end{array}\right\}$$

$$\Rightarrow \quad \boldsymbol{u}\in C([0,T];\boldsymbol{H}).$$

(b) The weak solution $\boldsymbol{u}$ given by Theorem 68 is unique.

Let $\boldsymbol{u}_1,\boldsymbol{u}_2$ be two solutions of (1). Then the difference $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$ satisfies

$$\begin{aligned} &\boldsymbol{u}_t+\nu\boldsymbol{Au}+\boldsymbol{Bu}_1-\boldsymbol{Bu}_2=0, \\ &\boldsymbol{u}(0)=0. \end{aligned} \tag{13}$$

Acting both sides of (13) to $\boldsymbol{u}$, and invoking Lemma 57, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2+\nu\|\nabla\boldsymbol{u}\|_2^2 &= -b(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u})+b(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}) \\ &= -b(\boldsymbol{u}_1,\boldsymbol{u},\boldsymbol{u})-b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}) \\ &= -b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}) \\ &\leq 2^{1/2}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}_2\|_2 \\ &\leq \nu\|\nabla\boldsymbol{u}\|_2^2+\frac{1}{2\nu}\|\nabla\boldsymbol{u}_2\|_2^2\|\boldsymbol{u}\|_2^2. \end{aligned}$$

Thus

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2\leq\frac{1}{\nu}\|\nabla\boldsymbol{u}_2\|_2^2\|\boldsymbol{u}\|_2^2,$$

$$\|\boldsymbol{u}(t)\|_2^2 \leq \|\boldsymbol{u}(0)\|_2^2 \, e^{\displaystyle\frac{1}{\nu}\int_0^t \|\nabla \boldsymbol{u}_2(s)\|_2^2 \, ds} = 0.$$

Thus $\boldsymbol{u}_1 = \boldsymbol{u}_2$, and the solution is unique.

□

Remark 72. *(a) By Lemma 69, we have $\boldsymbol{u} \in \boldsymbol{L}^4(Q)$, if $n = 2$.*

(b) The case Ω is unbounded is treated in the same way as Theorem 71.

10.4. **On regularity and uniqueness ($n = 3$)**

The 3D counterpart of Lemma 69 is the following

Lemma 73. *If $n = 3$, for any open bounded Ω,*

$$\|v\|_{L^4(\Omega)} \leq 2^{1/2} \|v\|_{L^2(\Omega)}^{1/4} \|\nabla v\|_{L^2(\Omega)}^{3/4}, \; v \in H_0^1(\Omega). \tag{14}$$

Proof. We only have to prove (14) for $v \in C_c^1(\Omega)$. For such a v, we calculate as

$$\begin{aligned}
\int_\Omega |v|^4 \, dx &= \int_{\boldsymbol{R}^3} |v|^4 \, dx \\
&= \int_{\boldsymbol{R}} \left(\int_{\boldsymbol{R}^2} |v|^4 \, dx_1 dx_2 \right) dx_3 \\
&\leq 2 \int_{\boldsymbol{R}} \left(\int_{\boldsymbol{R}^2} |v|^2 \, dx_1 dx_2 \right) \cdot \left(\int_{\boldsymbol{R}^2} |\nabla_h v|^2 \, dx_1 dx_2 \right) dx_3 \\
&\quad (\nabla_h = (\partial_1, \partial_2) \text{ is the horizontal gradient}) \\
&\leq 2 \sup_{x_3} \int_{\boldsymbol{R}^2} |v|^2 \, dx_1 dx_2 \cdot \int_{\boldsymbol{R}^3} |\nabla_h v|^2 \, dx \\
&\leq 2 \int_{\boldsymbol{R}^2} \sup_{x_3} |v|^2 \, dx_1 dx_2 \cdot \|\nabla_h v\|_2^2 \\
&\leq 2^2 \int_{\boldsymbol{R}^3} |v| \, |\partial_3 v| \, dx \cdot \|\nabla_h v\|_2^2 \\
&\leq 2^2 \|v\|_2 \|\partial_3 v\|_2 \|\nabla_h v\|_2^2 \\
&\leq 2^2 \|v\|_2 \|\nabla v\|_2^3 \\
&= 2^2 \|v\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}^3 .
\end{aligned}$$

□

Theorem 74. *If $n = 3$, the solution $\boldsymbol{u}$ of (1) given by Theorem 68 satisfies*

$$\boldsymbol{u} \in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega)),\ \boldsymbol{u}' \in L^{4/3}(0,T;\boldsymbol{V}').$$

Proof. 1. $\boldsymbol{u} \in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega))$.

By Lemma 73,

$$\|\boldsymbol{u}\|_4 \leq 2^{1/2}\,\|\boldsymbol{u}\|_2^{1/4}\,\|\nabla\boldsymbol{u}\|_2^{3/4} \in L^{8/3}(0,T).$$

2. $\boldsymbol{u}' \in L^{4/3}(0,T;\boldsymbol{V}')$.

This is already proved in the proof of Lemma 70, see (12).

□

Theorem 75. *If $n = 3$, there is at most one weak solution of (1) such that*

$$\boldsymbol{u} \in L^8(0,T;\boldsymbol{L}^4(\Omega)). \tag{15}$$

Such a solution belongs to $C([0,T];\boldsymbol{H})$.

Proof. 1. $\boldsymbol{Bu} \in L^2(0,T;\boldsymbol{V}')$.

In fact,

$$\begin{aligned} &\langle \boldsymbol{Bu},\boldsymbol{v}\rangle = b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = -b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{u}) \leq \|\boldsymbol{u}\|_4^2\,\|\nabla\boldsymbol{v}\|_2 \\ \Rightarrow\ &\|\boldsymbol{Bu}\|_{\boldsymbol{V}'} \leq \|\boldsymbol{u}\|_4^2 \in L^4(0,T) \subset L^2(0,T). \end{aligned}$$

2. $\boldsymbol{u}' = \boldsymbol{f} - \nu\boldsymbol{Au} - \boldsymbol{Bu} \in L^2(0,T;\boldsymbol{V}')$.
3. $\boldsymbol{u} \in C([0,T],\boldsymbol{H})$ follows from Lemma 57, as done several times before.
4. Proof of uniqueness.

Let $\boldsymbol{u}_1,\boldsymbol{u}_2$ be two weak solutions of (1) satisfying (15) (in fact only $\boldsymbol{u}_1$ or $\boldsymbol{u}_2$ satisfying (15) is enough to conclude the uniqueness). Then the difference $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ satisfies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \nu\,\|\nabla\boldsymbol{u}\|_2^2 &= -b(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}) + b(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}) \\ &= -b(\boldsymbol{u}_1,\boldsymbol{u},\boldsymbol{u}) - b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}) \\ &= -b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}) \end{aligned}$$

$$
\begin{aligned}
&= b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_2)\\
&\leq \|\boldsymbol{u}\|_4 \|\nabla \boldsymbol{u}\|^2 \|\boldsymbol{u}_2\|_4^2\\
&\leq 2^{1/2} \|\boldsymbol{u}\|_2^{1/4} \|\nabla \boldsymbol{u}\|_2^{7/4} \|\boldsymbol{u}_2\|_4\\
&\leq \nu \|\nabla \boldsymbol{u}(t)\|_2^2 + \frac{1}{2\nu} \|\boldsymbol{u}_2\|_4^8 \|\boldsymbol{u}\| 2^2.
\end{aligned}
$$

Thus

$$
\frac{d}{dt}\|u\|_2^2 \leq \frac{1}{\nu}\|\boldsymbol{u}_2\|_4^8 \|\boldsymbol{u}\|_2^2,
$$

$$
\|\boldsymbol{u}\|_2^2 \leq \|\boldsymbol{u}(0)\|_2^2 e^{\frac{1}{\nu}\int_0^t \|\boldsymbol{u}_2(s)\|_4^8 \, ds} = 0.
$$

And hence $\boldsymbol{u}_1 = \boldsymbol{u}_2$, the weak solution satisfying (15) is unique.

□

Remark 76. *1. The case Ω is unbounded is treated in the same way.*

2. Serrin-type uniqueness criteria for general n.

Let $\boldsymbol{u} \in L^r(0,T;\boldsymbol{L}^s(\Omega))$ is a weak solution of (1) with

$$
\frac{2}{r} + \frac{n}{s} \begin{cases} \leq 1, & \text{if } \Omega \text{ is bounded},\\ = 1, & \text{if } \Omega \text{ is unbounded}.\end{cases}
$$

Then $\boldsymbol{u}$ is the only weak solution of (1).

10.5. More regular solutions

10.5.1. *The 2D case*

Theorem 77. *We assume that $n = 2$ and that*

$$
\boldsymbol{f}, \boldsymbol{f}' \in L^2(0,T;\boldsymbol{V}'),\ \boldsymbol{f}(0) \in \boldsymbol{H};
$$
$$
\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}.
$$

The the unique solution of (1) given by Theorems 68 and 71 satisfies

$$
\boldsymbol{u}' \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}). \tag{16}
$$

Proof. We just need to do a priori estimates.

By (3), we have

$$\boldsymbol{u}' + \nu \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f},$$

that is,

$$\langle \boldsymbol{u}', \boldsymbol{v}\rangle + \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + b\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}\right) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle\,, \ \boldsymbol{v} \in \boldsymbol{V}. \tag{17}$$

Differentiating (17) yields

$$\langle \boldsymbol{u}'', \boldsymbol{v}\rangle + \nu\left(\nabla \boldsymbol{u}', \nabla \boldsymbol{v}\right) + b\left(\boldsymbol{u}', \boldsymbol{u}, \boldsymbol{v}\right) + b\left(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{v}\right) = \langle \boldsymbol{f}', \boldsymbol{v}\rangle\,.$$

Taking $\boldsymbol{v} = \boldsymbol{u}'$ in the above equality, we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{u}'\right\|_2^2 + \nu\left\|\nabla \boldsymbol{u}'\right\|_2^2 \\
= \ & -b\left(\boldsymbol{u}', \boldsymbol{u}, \boldsymbol{u}'\right) + \langle \boldsymbol{f}', \boldsymbol{u}'\rangle \\
\leq \ & 2^{1/2}\left\|\boldsymbol{u}'\right\|_2\left\|\nabla \boldsymbol{u}'\right\|_2\left\|\nabla \boldsymbol{u}\right\|_2 + \left\|\boldsymbol{f}'\right\|_{V'}\left\|\nabla \boldsymbol{u}'\right\|_2 \ \text{(by (9))} \\
\leq \ & \left[\frac{\nu}{4}\left\|\nabla \boldsymbol{u}'\right\|_2^2 + \frac{2}{\nu}\left\|\boldsymbol{u}'\right\|_2^2\left\|\nabla \boldsymbol{u}\right\|_2^2\right] + \left[\frac{\nu}{4}\left\|\nabla \boldsymbol{u}'\right\|_2^2 + \frac{1}{\nu}\left\|\boldsymbol{f}'\right\|_{\boldsymbol{V}'}^2.\right]
\end{aligned} \tag{18}$$

Thus

$$\frac{d}{dt}\left\|\boldsymbol{u}'\right\|_2^2 + \nu\left\|\nabla \boldsymbol{u}'\right\|_2^2 \leq \frac{2}{\nu}\left\|\boldsymbol{u}'\right\|_2^2\left\|\nabla \boldsymbol{u}\right\|_2^2 + \frac{2}{\nu}\left\|\boldsymbol{f}'\right\|_{\boldsymbol{V}'}^2.$$

Gronwall inequality and the fact that $\boldsymbol{u} \in L^2(0, T; \boldsymbol{H})$ yield (16) as desired. □

Remark 78. *1. In fact, in (18), we have used the Poincaré inequality, and for simplicity, we omit the constant (just using the equivalent norm).*

2. *When applying Gronwall inequality, we have in fact used* $\boldsymbol{u}'(0) \in \boldsymbol{H}$*. Indeed,*

$$\boldsymbol{u}'(0) = \boldsymbol{f}(0) - \nu \boldsymbol{A}\boldsymbol{u}(0) - \boldsymbol{B}\boldsymbol{u}(0);$$

$$\|\boldsymbol{u}'(0)\|_2 \leq \|\boldsymbol{f}(0)\|_2 + \nu \|\boldsymbol{A}\boldsymbol{u}(0)\|_2 + \|\boldsymbol{B}\boldsymbol{u}(0)\|_2 ;$$

$$\begin{aligned} \|\boldsymbol{A}\boldsymbol{u}(0)\|_2 &= \sup_{\|\boldsymbol{v}\|_2=1} |\langle \boldsymbol{A}\boldsymbol{u}(0), \boldsymbol{v}\rangle| \\ &= \sup_{\|\boldsymbol{v}\|_2=1} |(\nabla \boldsymbol{u}(0), \nabla \boldsymbol{v})| \\ &= \sup_{\|\boldsymbol{v}\|_2=1} |(\Delta \boldsymbol{u}(0), \boldsymbol{v})| \\ &\leq \|\Delta \boldsymbol{u}(0)\|_2 \\ &\leq \|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} ; \\ \|\boldsymbol{B}\boldsymbol{u}(0)\|_2 &= \sup_{\|\boldsymbol{v}\|_2=1} |[\boldsymbol{B}\boldsymbol{u}(0), \boldsymbol{v}]| \\ &= \sup_{\|\boldsymbol{v}\|_2=1} |b(\boldsymbol{u}(0), \boldsymbol{u}(0), \boldsymbol{v})| \\ &\leq \sup_{\|\boldsymbol{v}\|_2=1} \|\boldsymbol{u}(0)\|_4 \|\nabla \boldsymbol{u}(0)\|_4 \|\boldsymbol{v}\|_2 \\ &\leq 2^{1/2} \|\boldsymbol{u}(0)\|_2^{1/2} \|\nabla \boldsymbol{u}(0)\|_2^{1/2} \|\nabla^2 \boldsymbol{u}(0)\|_2^{1/2} \\ &\leq 2^{1/2} \|\boldsymbol{u}(0)\|_{\boldsymbol{H}^2(\Omega)}^2 . \end{aligned} \tag{19}$$

Theorem 79. *The assumptions are those of Theorem 77, and we assume moreover that* Ω *is a bounded set of class* C^2 *and that*

$$\boldsymbol{f} \in L^\infty(0, T; \boldsymbol{H}).$$

Then

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}^2(\Omega)).$$

Proof. (3) and Proposition 7 yield some distribution p satisfying

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} - \boldsymbol{u}' - \boldsymbol{u} \cdot \nabla \boldsymbol{u}. \tag{20}$$

We may then bootstrap the regularity of $\boldsymbol{u}$ by invoking Proposition 29, Chapter 1.

In fact, we have already that

$$\boldsymbol{f} \in L^\infty(0, T; \boldsymbol{H}),$$

and by Theorem 77 that

$$\boldsymbol{u}' \in L^\infty(0, T; \boldsymbol{H}).$$

Thus

$$\begin{aligned}
& \boldsymbol{u}(t)=\boldsymbol{u}(0)+\int_0^t \boldsymbol{u}'(s)ds \\
\Rightarrow\quad & \nabla\boldsymbol{u}(t)=\nabla\boldsymbol{u}(0)+\int_0^t \nabla\boldsymbol{u}'(s)ds \\
\Rightarrow\quad & \left\{\begin{array}{lll} \|\boldsymbol{u}(t)\|_2 & \le & \|\nabla\boldsymbol{u}(0)\|_2+\int_0^t \|\nabla\boldsymbol{u}'(s)\|_2\, ds \\ & \le & \|\nabla\boldsymbol{u}(0)\|_2+\|\nabla\boldsymbol{u}'\|_{L^2(0,t;L^2(\Omega))}\, t^{1/2} \\ & \in & L^\infty(0,T) \end{array}\right. \\
\Rightarrow\quad & \boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^1(\Omega)) \\
\Rightarrow\quad & \boldsymbol{u}\cdot\nabla\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{L}^{4/3}(\Omega))\ \ \text{(by Hölder inequality)} \\
\Rightarrow\quad & \boldsymbol{u}\in L^\infty(0,T;\boldsymbol{W}^{2,4/3}(\Omega))\subset L^\infty(Q)\ \ \text{(by (20) and (6) in Chapter 1)} \\
\Rightarrow\quad & \boldsymbol{u}\cdot\nabla\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{L}^{2}(\Omega))\ \text{(by Hölder inequality again)} \\
\Rightarrow\quad & \boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^2(\Omega))\ \text{(by (20) and (6) in Chapter 1 again)}\,.
\end{aligned}$$

□

Remark 80. *As in Proposition 49, Chapter 1, we may have intermediate regularity properties of* $\mathbf{u}$ *for suitable hypotheses on the data. Moreover,*

$$\left.\begin{array}{l} \Omega\in C^\infty \\ \boldsymbol{u}_0\in C^\infty\left(\bar\Omega\right) \\ \boldsymbol{f}\in C^\infty\left(\bar Q\right) \end{array}\right\}\Rightarrow \boldsymbol{u}\in C^\infty\left(\bar Q\right).$$

10.5.2. *The 3D case*

Theorem 81. *We assume that* $n=3$ *and that given* $\boldsymbol{f}$ *and* $\boldsymbol{u}_0$ *satisfying*

$$\boldsymbol{u}_0\in\boldsymbol{H}^2(\Omega)\cap\boldsymbol{V},$$
$$\boldsymbol{f}\in L^\infty(0,T;\boldsymbol{H}),\ \ \boldsymbol{f}'\in L^1(0,T;\boldsymbol{H}),$$

with a "smallness" condition:

$$\nu^3>16C\left(F+B\max\{G,1\}\right)\gamma_\Omega, \tag{21}$$

where

1. $C = \|\boldsymbol{u}\|_2 + \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}$;
2. $F = \|\boldsymbol{f}\|_{L^\infty(0,T;L^2(\Omega))}$;
3. $B = \|\boldsymbol{f}\|_{L^\infty(0,T;L^2(\Omega))} + \nu\,\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} + 2^{1/2}\,\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^2(\Omega)}$;
4. $G = e^{\|\boldsymbol{f}'\|_{L^1(0,T;\boldsymbol{L}^2(\Omega))}}$;
5. γ_Ω *is the Poincaré constant, i.e.*

$$\gamma_\Omega = \inf\left\{\gamma \geq 0;\ \|v\|_2 \leq \gamma\,\|\nabla v\|_2\,,\ v \in H_0^1(\Omega)\right\}.$$

Then there exists an unique solution of (1) *which satisfies moreover*

$$\boldsymbol{u}' \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}). \tag{22}$$

Proof. 1. Once (22) is shown, uniqueness of solutions follows from Theorem 75 immediately.

Indeed,

$$\begin{aligned}\boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}) \ &\Rightarrow\ \boldsymbol{u} \in C([0,T];\boldsymbol{V})\\ &\Rightarrow\ \boldsymbol{u} \in L^\infty(0,T;\boldsymbol{L}^4(\Omega)) \subset L^8(0,T;\boldsymbol{L}^4(\Omega)).\end{aligned}$$

2. As we did in the course of proof of Theorem 77, in order to prove (22), we need only do a priori estimates.

Recalling (18), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\|\boldsymbol{u}'\|_2^2 + \nu\,\|\nabla\boldsymbol{u}'\|_2^2 &= -b\,(\boldsymbol{u}',\boldsymbol{u},\boldsymbol{u}') + \langle\boldsymbol{f}',\boldsymbol{u}'\rangle\\ &= b\,(\boldsymbol{u}',\boldsymbol{u}',\boldsymbol{u}) + \langle\boldsymbol{f}',\boldsymbol{u}'\rangle\\ &\leq \|\boldsymbol{u}'\|_4\,\|\nabla\boldsymbol{u}'\|_2\,\|\boldsymbol{u}\|_4 + \|\boldsymbol{f}'\|_2\,\|\boldsymbol{u}'\|_2\\ &\leq 2\,\|\boldsymbol{u}'\|_2^{1/4}\,\|\nabla\boldsymbol{u}'\|_2^{7/4}\,\|\boldsymbol{u}\|_2^{1/4}\,\|\nabla\boldsymbol{u}\|_2^{3/4} + \|\boldsymbol{f}'\|_2\,\|\boldsymbol{u}'\|_2\\ &\leq 2\gamma_\Omega^{1/2}\,\|\nabla\boldsymbol{u}'\|_2^2\,\|\nabla\boldsymbol{u}\|_2 + \|\boldsymbol{f}'\|_2\,\|\boldsymbol{u}'\|_2\,.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\,\|\boldsymbol{u}'\|_2^2 + \left[\nu - 2\gamma_\Omega^{1/2}\,\|\nabla\boldsymbol{u}\|_2\right]\|\nabla\boldsymbol{u}'\|_2^2 \leq \|\boldsymbol{f}'\|_2\,\|\boldsymbol{u}'\|_2\,. \tag{23}$$

3. Let us first estimate $\|\boldsymbol{u}_0'\|_2$.

   Due to (19),

$$\|\boldsymbol{u}_0'\|_2 \leq \|\boldsymbol{f}'(0)\|_2 + \nu \|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} + 2^{1/2} \|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)}^2 \equiv B. \tag{24}$$

4. Thus if

$$S(t) \equiv \nu - 2\gamma_\Omega^{1/2} \|\nabla \boldsymbol{u}\|_2 > \frac{\nu}{2},\ t \in [0,T], \tag{25}$$

   then applying Gronwall inequality to (23) yields (22) as desired.

5. To prove (25), we first bound $\|\nabla \boldsymbol{u}\|_2$ in terms of $\|\boldsymbol{u}'\|_2$.

   In fact, taking the inner product of (1) with $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{u}\|_2^2 + \nu \|\nabla \boldsymbol{u}\|_2^2 = \langle \boldsymbol{f}, \boldsymbol{u}\rangle . \tag{26}$$

   (a) Integrating (26) yields

$$\begin{aligned} \|\boldsymbol{u}\|_2 \tfrac{d}{dt} \|\boldsymbol{u}\|_2 &\leq \|\boldsymbol{f}\|_2 \|\boldsymbol{u}\|_2 , \\ \tfrac{d}{dt} \|\boldsymbol{u}\|_2 &\leq \|\boldsymbol{f}\|_2 , \\ \|\boldsymbol{u}\|_2 &\leq \|\boldsymbol{u}_0\|_2 + \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} T \equiv C. \end{aligned}$$

   (b) Invoking Lemma 57 gives

$$\begin{aligned} \nu \|\nabla \boldsymbol{u}\|_2^2 &= \langle \boldsymbol{f}, \boldsymbol{u}\rangle - \langle \boldsymbol{u}', \boldsymbol{u}\rangle \\ &\leq \|\boldsymbol{f}\|_2 \|\boldsymbol{u}\|_2 + \|\boldsymbol{u}'\|_2 \|\boldsymbol{u}\|_2 \\ &\leq C\left(F + \|\boldsymbol{u}'\|_2\right). \end{aligned} \tag{27}$$

6. Then we show

$$S(0) > \frac{\nu}{2}. \tag{28}$$

   In fact,

$$\begin{aligned} S(0) &= \nu - 2\gamma_\Omega^{1/2} \|\nabla \boldsymbol{u}_0\|_2 \\ &\geq \nu - 2\gamma_\Omega^{1/2} \left[\frac{C\left(F + \|\boldsymbol{u}_0'\|_2\right)}{\nu}\right]^{1/2} \quad \text{(by (27))} \\ &\leq \nu - 2\left[\frac{C\left(F + B\right)\gamma_\Omega}{\nu}\right]^{1/2} \quad \text{(by (24))} \end{aligned}$$

$$> \frac{\nu}{2} \text{ (by (21))}.$$

7. And finally we argue by contradiction to verify (25).

   Suppose that there is some $t_* \in [0,T]$ such that $S(t_*) \le \frac{\nu}{2}$. Take

$$\tau = \min\left\{t \in [0,T];\ S(t) = \frac{\nu}{2}\right\},$$

then (argue by invoking the intermediate value theorem for continuous functions)

$$\begin{aligned} S(t) &> \tfrac{\nu}{2},\ 0 \le t < \tau, \\ S(\tau) &= \tfrac{\nu}{2}. \end{aligned} \tag{29}$$

Applying Gronwall inequality to (23) on $[0,\tau]$ yields

$$\|\boldsymbol{u}'(\tau)\|_2 \le \|\boldsymbol{u}_0'\|_2\, G \le BG \text{ (by (24))}.$$

Thus (27) implies that

$$\begin{aligned} S(\tau) &= \nu - 2\gamma_\Omega^{1/2}\, \|\nabla \boldsymbol{u}(\tau)\|_2 \\ &\ge \nu - 2\left[\frac{C\,(F+BG)\,\gamma_\Omega}{\nu}\right]^{1/2} \\ &> \frac{\nu}{2} \text{ (by (21))}, \end{aligned}$$

which contradicts $(29)_2$. This completes the proof of Theorem 81.

□

Theorem 82. *With the assumption of Theorem 81 and if we assume moreover that Ω is of class C^2, the function $\boldsymbol{u}$ satsifies*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^2(\Omega)). \tag{30}$$

Proof. The key point is the same as Theorem 79. But due to the critical Sobolev imbedding, we need to bootstrap three times. □

Remark 83. *The same remark about regularity as Remark 80 holds.*

10.5.3. *Introduction of the Pressure* $(n \leq 4)$

The existence of a pressure p follows from Propositions 7, 8. Meanwhile, the regularity properties of p can be deduced from Proposition 29.

10.6. **Relations between the problems of existence and uniqueness** $(n = 3)$

In this subsection, we call

1. $\boldsymbol{u}$ is a weak solution to (1) if it satisfies (2)-(4), and thus
   (a) by Theorem (74) that
   $$\boldsymbol{u} \in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega)),\ \boldsymbol{u}' \in L^{4/3}(0,T;\boldsymbol{V}');$$
   (b) by (**??**) that
   $$\|\boldsymbol{u}(t)\|_2^2 + 2\nu\int_0^t \|\nabla\boldsymbol{u}(s)\|_2^2\,ds \leq \|\boldsymbol{u}_0\|_2^2 + 2\int_0^t \langle \boldsymbol{f}(s),\boldsymbol{u}(s)\rangle\,ds,\ t\in[0,T];$$
   (c) according to Theorems 68, 71, we know the existence but not the uniqueness of weak solutions.
2. $\boldsymbol{v}$ is a strong solution if $\boldsymbol{v}$ is a weak solution satisfying furthermore
   $$\boldsymbol{v} \in L^8(0,T;\boldsymbol{L}^4(\Omega)),$$
   and thus
   (a) via Lemma 57, $\boldsymbol{v}$ satisfies the energy equality:
   $$\|\boldsymbol{v}(t)\|_2^2 + 2\nu\int_0^t \|\nabla\boldsymbol{v}(s)\|_2^2\,ds = \|\boldsymbol{v}_0\|_2^2 + 2\int_0^t \langle \boldsymbol{f}(s),\boldsymbol{v}(s)\rangle\,ds,\ t\in[0,T].$$
   (b) according to Theorems 75, 81, we know the uniqueness but not the existence of strong solutions (except in some very restrictive case—see the ”smallness” condition (21)).

The problems of the uniqueness of weak solutions and of the existence of strong solutions are related as follows:

Theorem 84. *We assume that $n = 3$ and that $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are arbitrarily given,*

$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}),\ \boldsymbol{u}_0 \in \boldsymbol{H}.$$

If there exists a strong solution $\boldsymbol{v}$ *to* (1)*, then there does not exist any other weak solution* $\boldsymbol{u}$*.*

Proof. This is a standard "weak = strong" uniqueness result. For the proof, see [2] or [3]. □

10.7. Utilization of a special basis—strong solutions

Let $\Omega(\subset \boldsymbol{R}^n, n = 2, 3)$ be a bounded Lipschitz open set. We shall use a special basis for the Galerkin method—the basis of eigenfunctions of the Stokes problem (c.f. Subsection 2.6, Chapter 1) —to obtain further a priori estimates on the solution and existence results of regular solutions.

1. Preliminary results.

   Before we do this, let us give two preliminary results.

Lemma 85. *Let* Ω *be a bounded open set of class* C^2 *in* $\boldsymbol{R}^n$*. Then* $\|\boldsymbol{A}\boldsymbol{u}\|_2$ *is a norm on* $\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$*, which is equivalent to the norm induced by* $\boldsymbol{H}^2(\Omega)$*.*

Proof. This is a simple consequence of Proposition 29, Chapter 1. □

Lemma 86. *Assume that* $\Omega(\subset \boldsymbol{R}^n, n = 2, 3)$ *is bounded and of class* C^2*. If* $\boldsymbol{u} \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$*, then* $\boldsymbol{B}\boldsymbol{u} \in \boldsymbol{H} \subset \boldsymbol{L}^2(\Omega)$*, and*

$$\|\boldsymbol{B}\boldsymbol{u}\|_2 \le C_2 \|\boldsymbol{u}\|_2^{1/2} \|\nabla \boldsymbol{u}\|_2 \|\boldsymbol{A}\boldsymbol{u}\|_2^{1/2}, \text{ if } n = 2, \tag{31}$$

$$\|\boldsymbol{B}\boldsymbol{u}\|_2 \le C_3 \|\nabla \boldsymbol{u}\|_2^{3/2} \|\boldsymbol{A}\boldsymbol{u}\|_2^{1/2}, \text{ if } n = 3. \tag{32}$$

Proof. (a) The case $n = 2$.

$$\begin{aligned}
\|\boldsymbol{B}\boldsymbol{u}\|_2 &= \sup_{\|\boldsymbol{v}\|_2=1} |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| \\
&\le \sup_{\|\boldsymbol{v}\|_2=1} \|\boldsymbol{u}\|_4 \|\nabla \boldsymbol{u}\|_4 \|\boldsymbol{v}\|_2 \\
&\le 2^{1/2} \|\boldsymbol{u}\|_2^{1/2} \|\nabla \boldsymbol{u}\|_2 \left\|\nabla^2 \boldsymbol{u}\right\|_2^{1/2} \text{ (by Lemma 69)} \\
&\le C_2 \|\boldsymbol{u}\|_2^{1/2} \|\nabla \boldsymbol{u}\|_2^{1/2} \|\boldsymbol{A}\boldsymbol{u}\|_2^{1/2} \text{ (by Lemma 85)}.
\end{aligned}$$

(b) The case $n=3$.

$$\begin{aligned}
\|\boldsymbol{B}\boldsymbol{u}\|_2 &= \sup_{\|\boldsymbol{v}\|_2=1} |b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})| \\
&\leq \sup_{\|\boldsymbol{v}\|_2=1} \int |\boldsymbol{u}| \cdot |\nabla \boldsymbol{u}|^{1/2} \cdot |\nabla \boldsymbol{u}|^{1/2} \cdot |\boldsymbol{v}| \\
&\leq \sup_{\|\boldsymbol{v}\|_2=1} \|\boldsymbol{u}\|_6 \|\nabla \boldsymbol{u}\|_2^{1/2} \|\nabla \boldsymbol{u}\|_6^{1/2} \|\nabla \boldsymbol{v}\|_2 \\
&\leq C_3 \|\nabla \boldsymbol{u}\|_2^{3/2} \|\boldsymbol{A}\boldsymbol{u}\|_2^{1/2} \text{ (by Lemma 85)}.
\end{aligned}$$

□

2. The 2D case.

Theorem 87. *We assume that Ω is a bounded open set of class C^2 in $\boldsymbol{R}^2$. Let $\boldsymbol{f}$ and $\boldsymbol{u}_0$ be given such that*

$$\boldsymbol{u}_0 \in \boldsymbol{H},\ \boldsymbol{f} \in L^2(0,T;\boldsymbol{H}).$$

Then there exists an unique solution to (1), which satisfies moreover

$$\sqrt{t}\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{H}^2(\Omega)), \tag{33}$$

$$\sqrt{t}\boldsymbol{u}' \in L^2(0,T;\boldsymbol{H}). \tag{34}$$

If $\boldsymbol{u}_0 \in \boldsymbol{V}$, then

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{H}^2(\Omega)), \tag{35}$$

$$\boldsymbol{u}' \in L^2(0,T;\boldsymbol{H}). \tag{36}$$

Proof. (a) The case $\boldsymbol{u}_0 \in \boldsymbol{H}$.

By (3),

$$\boldsymbol{u}' + \nu \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}. \tag{37}$$

Taking the inner product of (37) with $\boldsymbol{Au}$ in $\boldsymbol{L}^2(\boldsymbol{R}^2)$ (**it is here that we use the special basis**), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|_2^2 + \nu \|\boldsymbol{Au}\|_2^2 \\ \leq\ & \|\boldsymbol{Bu}\|_2 \|\boldsymbol{Au}\|_2 + \|\boldsymbol{f}\|_2 \|\boldsymbol{Au}\|_2 \\ \leq\ & C_2 \|\boldsymbol{u}\|_2^{1/2} \|\nabla \boldsymbol{u}\|_2 \|\boldsymbol{Au}\|_2^{3/2} + \|\boldsymbol{f}\|_2 \|\boldsymbol{Au}\|_2 \ \text{(by (31))} \\ \leq\ & \frac{3C_2}{4\nu} \|\boldsymbol{u}\|_2^2 \|\nabla \boldsymbol{u}\|_2^4 + \frac{1}{\nu}\|\boldsymbol{f}\|_2^2 + \frac{\nu}{2}\|\boldsymbol{Au}\|_2^2 . \end{aligned} \tag{38}$$

Thus

$$\begin{aligned} & \frac{d}{dt}\left[t\|\nabla \boldsymbol{u}\|_2^2\right] + \nu t \|\boldsymbol{Au}\|_2^2 \\ \leq\ & \|\nabla \boldsymbol{u}\|_2^2 + \frac{t}{\nu}\|\boldsymbol{f}\|_2^2 + \frac{3C_2}{2\nu}\left[\|\boldsymbol{u}\|_2^2 \|\nabla \boldsymbol{u}\|_2^2\right]\left[t\|\nabla \boldsymbol{u}\|_2^2\right]. \end{aligned}$$

Gronwall inequality together with the fact (2) yields (33) (**it is the initial data that we need multiplying** t). (34) readily follows from

$$\sqrt{t}\boldsymbol{u}' = \sqrt{t}\left(\boldsymbol{f} - \nu \boldsymbol{Au} - \boldsymbol{Bu}\right) \in L^2(0,T;\boldsymbol{H}).$$

(b) The case $\boldsymbol{u}_0 \in \boldsymbol{V}$.

In this case, we can apply Gronwall inequality directly to (38), yielding (35). Meanwhile, (36) follows exactly the same as the case when $\boldsymbol{u}_0 \in \boldsymbol{H}$.

□

3. The 3D case.

Theorem 88. *We assume that* Ω *is a bounded open set of class* C^2 *in* $\boldsymbol{R}^2$*. Let* $\boldsymbol{f}$ *and* $\boldsymbol{u}_0$ *be given such that*

$$\boldsymbol{u}_0 \in \boldsymbol{V},\ \boldsymbol{f} \in L^\infty(0,T;\boldsymbol{H}).$$

Then there exists a $T^* = \min\{T, T_1\}$ *with*

$$\begin{aligned} T_1 = & \frac{\nu}{2\max\{3C_3, 4\}\left(\|\nabla \boldsymbol{u}_0\|_2^2 + F^{2/3}\right)^2} \\ & \left(F = \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\boldsymbol{R}^2))}, C_3 \in (32)\right), \end{aligned} \tag{39}$$

such that there exists an unique solution $\boldsymbol{u}$ *of* (1) *on* $(0,T^*)$*; moreover* $\boldsymbol{u}$ *satisfies*

$$\boldsymbol{u}\in L^\infty(0,T^*;\boldsymbol{V})\cap L^2(0,T^*;\boldsymbol{H}^2(\Omega)), \tag{40}$$

$$\boldsymbol{u}'\in L^2(0,T^*;\boldsymbol{H}). \tag{41}$$

Proof. As in the proof of Theorem 87, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|_2^2+\nu\|\boldsymbol{A}\boldsymbol{u}\|_2^2 &\le \|\boldsymbol{B}\boldsymbol{u}\|_2\|\boldsymbol{A}\boldsymbol{u}\|_2+\|\boldsymbol{f}\|_2\|\boldsymbol{A}\boldsymbol{u}\|_2\\ &\le C_3\|\nabla\boldsymbol{u}\|_2^{3/2}\|\boldsymbol{A}\boldsymbol{u}\|_2^{3/2}+\|\boldsymbol{f}\|_2\|\boldsymbol{A}\boldsymbol{u}\|_2\\ &\le \frac{3C_3}{4\nu}\|\nabla\boldsymbol{u}\|_2^6+\frac{1}{\nu}\|\boldsymbol{f}\|_2^2+\frac{\nu}{2}\|\boldsymbol{A}\boldsymbol{u}\|_2^2,\end{aligned}$$

that is,

$$\frac{d}{dt}\|\nabla\boldsymbol{u}\|_2^2+\nu\|\boldsymbol{A}\boldsymbol{u}\|_2^2\le\frac{3C_3}{2\nu}\|\nabla\boldsymbol{u}\|_2^6+\frac{2}{\nu}\|\boldsymbol{f}\|_2^2. \tag{42}$$

Thus

$$\frac{d}{dt}\left[\|\nabla\boldsymbol{u}\|_2^2+F^{2/3}\right]\le\frac{1}{2\nu}\max\{3C_3,4\}\left[\|\nabla\boldsymbol{u}\|_2^2+F^{2/3}\right]^3.$$

Hence

$$\begin{aligned}\|\nabla\boldsymbol{u}\|_2^2+F^{2/3} &\le \frac{\left[\|\nabla\boldsymbol{u}_0\|_2^2+F^{2/3}\right]^2}{1-\frac{1}{\nu}\max\{3C_3,4\}\left[\|\nabla\boldsymbol{u}\|_2^2+F^{2/3}\right]^2t}\\ &\le 2\left[\|\nabla\boldsymbol{u}_0\|_2^2+F^{2/3}\right]^2\ \text{(by (39))}.\end{aligned}$$

The theorem then readily follows. □

10.8. Decay of solutions

We are going to show that for $\boldsymbol{f}=0$, the fluid tends to the equilibrium, as $t\to\infty$.

Theorem 89. *We assume that* Ω *is a* C^2 *open bounded set in* $\boldsymbol{R}^n(n=2,3)$*, and that* $\boldsymbol{u}_0\in\boldsymbol{V}$ *and* $\boldsymbol{f}=0$.

Then

$$\boldsymbol{u}\in\begin{cases}L^\infty(0,\infty,\boldsymbol{V}), & \text{if } n=2,\\ L^\infty(0,T_1,\boldsymbol{V})\cap L^\infty(T_2,\infty,\boldsymbol{V}), & \text{if } n=3.\end{cases}$$

Moreover, in both cases, $\mathbf{u}$ decays to 0 exponentially in $\boldsymbol{V}$ as $t \to \infty$.

Proof. Standard energy estimates show that

$$\|\boldsymbol{u}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}(s)\|_2^2\, ds \le \|\boldsymbol{u}_0\|_2^2\,,\ \forall\, t > 0, \tag{43}$$

and (38), (42) give

$$\frac{d}{dt}\|\nabla \boldsymbol{u}\|_2^2 + \frac{\nu}{\gamma_\Omega^2}\|\nabla \boldsymbol{u}\|_2^2 \le \frac{d}{dt}\|\nabla \boldsymbol{u}\|_2^2 + \nu\,\|\boldsymbol{A}\boldsymbol{u}\|_2^2 \le C\,\|\nabla \boldsymbol{u}\|_2^{2n}\,,$$

where

$$C = \begin{cases} \frac{3C_2}{2\nu}\,\|\boldsymbol{u}_0\|_2^2\,, & \text{if } n = 2,\\ \frac{3C_3}{2\nu}\,, & \text{if } n = 3,\end{cases}$$

and γ_Ω is the Poincaré constant.

Hence

$$\frac{d}{dt}\|\nabla \boldsymbol{u}\|_2^2 + \left[\frac{\nu}{\gamma_\Omega^2} - C\,\|\nabla \boldsymbol{u}\|_2^{2(n-1)}\right]\|\nabla \boldsymbol{u}\|_2^2 \le 0. \tag{44}$$

Now by (43), we can find some $T_2 \ge T_1 \in$ (39), such that

$$\|\nabla \boldsymbol{u}(T_2)\|_2^{2(n-1)} \le \frac{\nu}{2C\gamma_\Omega^2}.$$

And thus (44) yields

$$\frac{d}{dt}|_{t=T_2}\,\|\nabla \boldsymbol{u}(t)\|_2^2 + \frac{\nu}{2\gamma_\Omega^2}\|\nabla \boldsymbol{u}(T_2)\|_2^2 \le 0.$$

Then a simple argument that involves only differential calculus shows that

$$\begin{gathered}\|\nabla \boldsymbol{u}(t)\|_2^{2(n-1)} \le \frac{\nu}{2C\gamma_\Omega^2},\\ \frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|_2^2 + \frac{\nu}{2\gamma_\Omega^2}\|\nabla \boldsymbol{u}(t)\|_2^2 \le 0,\end{gathered} \tag{45}$$

for all $t \ge T_2$.

Notice that $(45)_2$ implies the exponential decay of $\boldsymbol{u}$ in $\boldsymbol{V}$ as desired. □

11. Rothe's approach to the existence of a weak solution

The reason that we need $n \leq 4$ in the existence results in Section 10 is two-folds.

1. In case $n \leq 4$, we have $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is a trilinear form on $\boldsymbol{H}_0^1(\Omega)$ $(\subset \boldsymbol{L}^n(\Omega))$, and thus $\boldsymbol{B}\boldsymbol{u} \in \boldsymbol{V}'$ is well-defined.
2. In case $n \leq 4$, we have furthermore

$$\begin{aligned} & \boldsymbol{B}\boldsymbol{u} \in L^{4/n}(0,T;\boldsymbol{V}') \subset L^1(0,T;\boldsymbol{V}') \ \text{(by (12))} \\ \Rightarrow \ & \boldsymbol{u}' \in L^1(0,T;\boldsymbol{V}') \\ \Rightarrow \ & \boldsymbol{u} \in C([0,T];\boldsymbol{V}') \\ \Rightarrow \ & \boldsymbol{u} \in C([0,T];\boldsymbol{H}_w) \ \text{(by Lemma 56)}\,. \end{aligned}$$

In this section, we shall use Rothe's approach to establish existence results for (1) in arbitrary space dimensions. A poem generated by the author can be found in [5].

11.1. The weak formulation

As remarked above, we need to re-state our problem, since $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w})$ is no longer well-defined on $\boldsymbol{H}_0^1(\Omega)^3$.

For this purpose, we introduce for each integer $s \geq 1$,

$$\boldsymbol{V}_s = \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in } \boldsymbol{H}^s(\Omega) \cap \boldsymbol{H}_0^1(\Omega).$$

Then we have properties of b and $\boldsymbol{B}$ as the following two lemmas show.

Lemma 90. *The form b is trilinear continuous on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}_s$ if $s \geq n/2$, and*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{v}\|_2 \|\boldsymbol{w}\|_{\boldsymbol{V}_s}\,. \tag{1}$$

Proof.

$$\begin{aligned} |b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| &= |b\,(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})| \\ &\leq \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{w}\|_n \|\nabla \boldsymbol{v}\|_{\frac{2n}{n-2}} \\ &\leq C \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{w}\|_{\boldsymbol{H}^{s-1}(\Omega)} \|\nabla \boldsymbol{v}\|_2 \end{aligned}$$

$$\left(-(s-1)+\frac{n}{2}\leq\frac{n}{n}\right).$$

□

Lemma 91. *If* $\boldsymbol{u}\in L^{\infty}(0,T;\boldsymbol{H})\cap L^{2}(0,T;\boldsymbol{V})$*, then* $\boldsymbol{B}\boldsymbol{u}\in L^{2}(0,T;\boldsymbol{V}_{s}')$ *for* $s\geq n/2$.

Proof.

$$\begin{aligned}
\|\boldsymbol{B}\boldsymbol{u}\|_{\boldsymbol{V}_s'} &= \sup_{\|\boldsymbol{v}\|_{\boldsymbol{V}_s}\leq 1}|b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})| \\
&\leq C\sup_{\|\boldsymbol{v}\|_{\boldsymbol{H}^s}\leq 1}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2\|\boldsymbol{v}\|_{\boldsymbol{H}^s} \\
&\leq \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2\in L^2(0,T).
\end{aligned}$$

□

We are now ready to give the weak formulation of (1) in arbitrary space dimensions.

Definition 92. *Given* $\boldsymbol{f}\in L^{2}(0,T;\boldsymbol{V}')$ *and* $\boldsymbol{u}_0\in\boldsymbol{H}$. *A measurable vector* $\boldsymbol{u}$ *defined on* $\Omega\times[0,T]$ *is said to be a weak solution of* (1) *if*

$$\begin{gathered}
\boldsymbol{u}\in C(0,T;\boldsymbol{H}_w)\cap L^2(0,T;\boldsymbol{V}), \\
\boldsymbol{u}'\in L^2(0,T;\boldsymbol{V}_s')\ (s\geq n/2)\,;
\end{gathered} \tag{2}$$

$$\boldsymbol{u}'+\nu\boldsymbol{A}\boldsymbol{u}+\boldsymbol{B}\boldsymbol{u}=\boldsymbol{f},\ \text{on } (0,T); \tag{3}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0. \tag{4}$$

The existence of such a weak solution is given by

Theorem 93. *Given* $\boldsymbol{f}\in L^{2}(0,T;\boldsymbol{V}')$ *and* $\boldsymbol{u}_0\in\boldsymbol{H}$. *Then there exists at least one weak solution* $\boldsymbol{u}$ *to* (1).

This Theorem is proved in Subsections 11.2,11.3,11.4, with the weak continuity in $\boldsymbol{H}$ of $\boldsymbol{u}$ a direct consequence of Lemma 56.

11.2. The approximate solutions

Let N be an integer which will later go to infinity and set $k = T/N$. For $m \in \{1, 2, \cdots, N\}$, we define

1.

$$\boldsymbol{f}^m = \frac{1}{k}\int_{(m-1)k}^{mk} \boldsymbol{f}(t)dt \in \boldsymbol{V}'; \tag{5}$$

2.

$$\begin{aligned} \boldsymbol{u}^0 &= \boldsymbol{u}_0, \\ \frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{k} + \nu \boldsymbol{A}\boldsymbol{u}^m + \boldsymbol{B}\boldsymbol{u}^m &= \boldsymbol{f}^m,\ m \geq 1. \end{aligned} \tag{6}$$

Here $\boldsymbol{u}^m$ depends on k, for simplicity, we denote it $\boldsymbol{u}^m$ in lieu of $\boldsymbol{u}_k^m$.

The existence of $\boldsymbol{u}^m \in (6)_2$ is asserted by

Lemma 94. *For each k and each $m \in \{1, 2, \cdots, N\}$, there exists at least one $\boldsymbol{u}^m$ satisfying $(6)_2$ and moreover*

$$\|\boldsymbol{u}^m\|_2^2 - \left\|\boldsymbol{u}^{m-1}\right\|_2^2 + \left\|\boldsymbol{u}^m - \boldsymbol{u}^{m-1}\right\|_2^2 + 2k\nu \left\|\nabla \boldsymbol{u}^m\right\|_2^2 \leq 2k \left\langle \boldsymbol{f}^m, \boldsymbol{u}^m \right\rangle. \tag{7}$$

Proof. Invoking Galerkin method, Lemma 41, Chapter 2, and the fact

$$2(\boldsymbol{a} - \boldsymbol{b}, \boldsymbol{a}) = \|\boldsymbol{a}\|_2^2 - \|\boldsymbol{b}\|_2^2 - \|\boldsymbol{a} - \boldsymbol{b}\|_2^2,\ \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{H},$$

we easily conclude the proof. □

Now we ready to define the approximate solution for each k (or N) as

$$\begin{aligned} &\boldsymbol{u}_k : [0, T] \to \boldsymbol{V},\ \boldsymbol{u}_k(t) = \boldsymbol{u}^m, t \in [(m-1)k, mk]; \\ &\boldsymbol{w}_k : [0, T] \to \boldsymbol{H}, \begin{cases} \boldsymbol{w}_k \text{ linear on } [(m-1)k, mk], \\ \boldsymbol{w}_k(mk) = \boldsymbol{u}^m. \end{cases} \end{aligned} \tag{8}$$

11.3. A priori estimates

Lemma 95.

$$\|\boldsymbol{u}^m\|_2^2 \le d_1, \tag{9}$$

$$k\sum_{m=1}^{N} \|\nabla \boldsymbol{u}^m\|_2^2 \le \frac{1}{\nu} d_1, \tag{10}$$

$$\sum_{m=1}^{N} \left\|\boldsymbol{u}^m - \boldsymbol{u}^{m-1}\right\|_2^2 \le d_1, \tag{11}$$

where

$$d_1 = \|\boldsymbol{u}_0\|_2^2 + \frac{1}{\nu}\int_0^t \|\boldsymbol{f}(s)\|_{\boldsymbol{V}'}^2\, ds. \tag{12}$$

Proof. This is a simple consequence of (7). □

Lemma 96.

$$k\sum_{k=1}^{N} \left\|\frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{k}\right\|_{\boldsymbol{V}'}^2 \text{ is uniformly bounded independent of } k.$$

Proof. This follows from $(6)_2$ and Lemma 91. □

Lemma 97. *The function $\boldsymbol{u}_k$ and $\boldsymbol{w}_k$ remain a bounded set of $L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$; $\boldsymbol{w}_k'$ is bounded in $L^2(0,T;\boldsymbol{V}_s')$ and*

$$\|\boldsymbol{u}_k - \boldsymbol{w}_k\|_{L^2(0,T;\boldsymbol{H})} \to 0, \text{ as } k \to 0. \tag{13}$$

Proof. The estimations of $\boldsymbol{u}_k$,$\boldsymbol{w}_k$,$\boldsymbol{w}_k'$ are just interpretations of (9), (10) and Lemma 96.

We need only show (13).

In fact,

$$\begin{aligned} \|\boldsymbol{u}_k - \boldsymbol{w}_k\|_{L^2(0,T;\boldsymbol{H})}^2 &= \int_0^T \|\boldsymbol{u}_k(t) - \boldsymbol{w}_k(t)\|_2^2\, dt \\ &= \sum_{m=1}^{N} \int_{(m-1)k}^{mk} \left\|\frac{t-mk}{k}\left(\boldsymbol{u}^m - \boldsymbol{u}^{m-1}\right)\right\|_2^2 dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^k \frac{s^2}{k^2} ds \cdot \sum_{m=1}^{N} \left\| \boldsymbol{u}^m - \boldsymbol{u}^{m-1} \right\|_2^2 \\
&= \frac{k}{3} \cdot \sum_{m=1}^{N} \left\| \boldsymbol{u}^m - \boldsymbol{u}^{m-1} \right\|_2^2 \\
&\leq \frac{k}{3} \cdot d_1 \ \text{(by (11))} \\
&\to 0, \ \text{as } k \to 0.
\end{aligned}
$$

□

11.4. **Passage to the limits**

Due to Lemma 97, we have, up to some subsequence, that

$$
\begin{aligned}
\boldsymbol{u}_k &\overset{*}{\rightharpoonup} \boldsymbol{u}, \quad \text{in } L^\infty(0,T;\boldsymbol{H}), \\
\boldsymbol{u}_k &\rightharpoonup \boldsymbol{u}, \quad \text{in } L^2(0,T;\boldsymbol{V});
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\boldsymbol{w}_k &\overset{*}{\rightharpoonup} \boldsymbol{u}_*, \quad \text{in } L^\infty(0,T;\boldsymbol{H}), \\
\boldsymbol{w}_k &\rightharpoonup \boldsymbol{u}_*, \quad \text{in } L^2(0,T;\boldsymbol{V});
\end{aligned} \tag{15}
$$

$$
\boldsymbol{w}_k' \rightharpoonup \boldsymbol{u}_*', \quad \text{in } L^2(0,T;\boldsymbol{V}_s'). \tag{16}
$$

Because of (13)(by $(14)_1$, $(15)_1$)), $\boldsymbol{u}_* = \boldsymbol{u}$.

Then Theorem 62 shows that (by $(15)_2$, (16))

$$
\boldsymbol{w}_k \to \boldsymbol{u}, \quad \text{in } L^2(0,T;\boldsymbol{H}). \tag{17}
$$

Thanks to (13),

$$
\boldsymbol{u}_k \to \boldsymbol{u}, \ \text{in } L^2(0,T;\boldsymbol{H}), \tag{18}
$$

also.

Now, $(6)_2$ can be interpreted as

$$
\boldsymbol{w}_k' + \nu \boldsymbol{A}\boldsymbol{u}_k + \boldsymbol{B}\boldsymbol{u}_k = \boldsymbol{f}_k, \tag{19}
$$

where

$$\boldsymbol{f}_k(t) = \boldsymbol{f}^m, \ (m-1)k \le t < mk.$$

We wish to pass to limit $k \to 0$ in (19).

Observe that

1. $\boldsymbol{A}\boldsymbol{u}_k \rightharpoonup \boldsymbol{A}\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{V}')$.

   This follows directly from (14).

2. $\boldsymbol{B}\boldsymbol{u} \rightharpoonup \boldsymbol{B}\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{V}_s)$.

   In view of Lemma 91, we need only verify, for $\forall\, \boldsymbol{v} \in C_c^\infty([0,T];C_c^\infty(\Omega))$, that

$$\begin{aligned}
\int_0^T \langle \boldsymbol{B}\boldsymbol{u}_k - \boldsymbol{B}\boldsymbol{u}, \boldsymbol{v}\rangle\, dt &= \int_0^T \left[b\left(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{v}\right) - b\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}\right)\right] dt \\
&= \int_0^T \left[b\left(\boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{u}_k, \boldsymbol{v}\right) + b\left(\boldsymbol{u}, \boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{v}\right)\right] dt \\
&= \int_0^T \left[b\left(\boldsymbol{u}_k - \boldsymbol{u}, \boldsymbol{u}_k, \boldsymbol{v}\right) - b\left(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_k - \boldsymbol{u}\right)\right] dt \\
&\Rightarrow 0 (\text{by } (18), (14)_2).
\end{aligned}$$

3. $\boldsymbol{f}_k \to \boldsymbol{f}$, in $L^2(0,T;\boldsymbol{V}')$.

   Due to the fact

$$\|\boldsymbol{f}_k\|_{L^2(0,T;\boldsymbol{V}')} \le \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{V}')},$$

we need only verify, for $\boldsymbol{f} \in C([0,T];\boldsymbol{V}')$, that

$$\begin{aligned}
\int_0^T \|\boldsymbol{f}_k - \boldsymbol{f}\|_{\boldsymbol{V}'}^2\, dt &= \sum_{m=1}^N \int_{(m-1)k}^{mk} \left\|\boldsymbol{f}(t) - \frac{1}{k}\int_{(m-1)k}^{mk} f(s)ds\right\|_{\boldsymbol{V}'} dt \\
&\le \sum_{m=1}^N \int_{(m-1)k}^{mk} \frac{1}{k}\int_{(m-1)k}^{mk} \|\boldsymbol{f}(s) - \boldsymbol{f}(t)\|_{\boldsymbol{V}'}\, dsdt \\
&\to 0, \text{ as } k \to 0.
\end{aligned}$$

We obtain by taking $k \to 0$ in (19) that

$$\boldsymbol{u}' + \nu \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}. \tag{20}$$

The last issue that we need to assure is that $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

For this purpose, we need

Lemma 98. *Suppose that X, Y be two Banach spaces satisfying*

$$X \subset Y,\ Y' \text{ is separable and dense in } X'.$$

Then

$$\left.\begin{array}{c} f_k \overset{*}{\rightharpoonup} f,\ \text{in } L^\infty(0,T;X) \\ \{\partial_t f_k\}_{k=1}^\infty \text{ is bounded in } L^p(0,T;X), 1<p\leq\infty \end{array}\right\} \Rightarrow f_k \to f \text{ in } C([0,T],X_w).$$

Proof. Let $\{\phi_m\}_{m=1}^\infty \subset Y'$ be dense in X', then the distance d in X_w is given by

$$d(g,h)=\sum_{m=1}^\infty \frac{1}{2^m}\frac{|\langle\phi_m, g-h\rangle|}{1+|\langle\phi_m,g-h\rangle|},\ g,h\in X.$$

To prove that $f_k \to f$ in $C([0,T];X_w)$, we invoke Arzela-Ascoli Theorem, and only need to show the equi-continuity as

$$\begin{aligned} d\left(f_k(t),f_k(s)\right) &= \sum_{m=1}^\infty \frac{1}{2^m}\frac{|\langle\phi_m,f_k(t)-f_k(s)\rangle|}{1+|\langle\phi_m,f_k(t)-f_k(s)\rangle|} \\ &= \left(\sum_{m=1}^M+\sum_{m=M+1}^\infty\right)\frac{1}{2^m}\frac{|\langle\phi_m,f_k(t)-f_k(s)\rangle|}{1+|\langle\phi_m,f_k(t)-f_k(s)\rangle|} \\ &\leq \sup_{1\leq m\leq M}|\langle\phi_m,f_k(t)-f_k(s)\rangle|+\frac{1}{2^M} \\ &= \sup_{1\leq m\leq M}\left|\left\langle\phi_m,\int_s^t\partial_\tau f_k(\tau)d\tau\right\rangle\right|+\frac{1}{2^M} \\ &= \sup_{1\leq m\leq M}\left|\int_s^t\langle\phi_m,\partial_\tau f_k(\tau)\rangle\,d\tau\right|+\frac{1}{2^M} \\ &\leq \sup_{1\leq m\leq M}\|\phi_m\|_{Y'}\cdot|t-s|^{1-\frac{1}{p}}\cdot\|\partial_t f_k\|_{L^p(0,T;Y)}+\frac{1}{2^M} \\ &\to 0,\ \text{as}\ \left(|t-s|\to 0,\ \text{then } M\to\infty\right). \end{aligned}$$

□

Now we conclude that $\boldsymbol{u}(0)=\boldsymbol{u}_0$. Indeed, $(15)_1$ and Lemma 97 together with Lemma 98 yield $\boldsymbol{u}_k \to \boldsymbol{u}$ in $C([0,T];\boldsymbol{H}_w)$. Thus

$$\boldsymbol{u}_0=\boldsymbol{w}_k(0)\to\boldsymbol{u}(0),\ \text{in } \boldsymbol{H}_w.$$

Remark 99. *The energy (in)equality can be easily deduced from* (7) *and Lemma* 57.

12. P.L. Lions

Pierre-Louis Lions (born August 11, 1956 in Grasse, Alpes-Maritimes) is a French mathematician. His parents were Jacques-Louis Lions, a mathematician and at that time professor at the University of Nancy, who in particular became President of the International Mathematical Union, and Andrée Olivier, his wife. He graduated from the école Normale Supérieure in 1977 (same year as Jean-Christophe Yoccoz). He received his doctorate from the University of Pierre and Marie Curie in 1979. Lions is listed as an **ISI highly cited researcher**.

He studies the theory of nonlinear partial differential equations, and received the Fields Medal for his mathematical work in 1994 while working at the University of Paris-Dauphine. Lions was the first to give a complete solution to the Boltzmann equation with proof. Other awards Lions received include the IBM Prize in 1987 and the Philip Morris Prize in 1991. He is a doctor honoris causa of Heriot-Watt University (Edinburgh) and of the City University of Hong-Kong. Currently, he holds the position of Professor of Partial differential equations and their applications at the prestigious Collége de France in Paris as well as a position at Ecole Polytechnique.

In the paper ”Viscosity solutions of Hamilton-Jacobi equations” (1983), written with Michael Crandall, he introduced the notion of viscosity solutions. This has had a great effect on the theory of partial differential equations.

This follows from P.L. Lions.

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