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RENORMALIZED NOTES FOR THE NAVIER-STOKES EQUATIONS IN 8TH NONLINEAR PDE SEMINARS AT XI'AN

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ABSTRACT. Introducing renormalized solutions, P.L. Lions succeeded in establishing the global existence of weak solutions for the isentropic compressible Navier-Stokes equations, while now I am writing these renormalized notes delivered by Professor G.P. Galdi from University of Pittsburgh to review previous knowledge and thank those professors and mates who we met at Xi'an.

CONTENTS

1. Steady-state Navier-Stokes BVP in a bounded domain 77
1.1. The formulation of the problem 77
1.2. Function spaces where we work in 78
1.3. Weak formulation of the problem 78
1.4. Garlerkin approximated solutions 79
1.5. Passage to limit as $N \to \infty$ 80
2. Steady-state Navier-Stokes BVP in an exterior domain 81
2.1. What is an exterior domain? 81
2.2. The problem we are interested in 81
2.3. Difficulties we encounter 81
2.4. Existence of a weak solution 82
3. Regularity of weak solutions 83
4. Steady-state body/liquid coupled system 85
4.1. The problem at hand 85
4.2. Difficulties we encounter 85

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4.3. Function spaces and Kohn's inequality 86
4.4. Weak formulation of the problem 88
4.5. Existence of a weak solution 89
4.6. Non-uniqueness, regularity and related topics 98
5. Time-dependent falling body problem—a review 99
6. Steady bifurcation theory of the Navier-Stokes problem in a bounded domain 100
6.1. What is a bifurcation? 100
6.2. Main result—functional/geometrical properties of the set of solutions 103
6.3. Abstract formulation of (1) 104
6.4. Properness property of M 104
6.5. Fredholm property of M 106
6.6. Degree property of M and the ending of Theorem 58 109
7. Exam at ∞ 113

1. Steady-state Navier-Stokes BVP in a bounded domain

1.1. The formulation of the problem

We consider

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \boldsymbol{v} = 0, \quad \text{on } \partial\Omega, \tag{1}$$

where

- $\Omega(\subset \boldsymbol{R}^n,\ n = 2, 3)$ is a bounded domain;
- $\boldsymbol{v}$ is the velocity, p is the pressure, $\boldsymbol{f}$ is the external force;
- $\nu > 0$ is the viscosity.

1.2. Function spaces where we work in

Denote by

$$\boldsymbol{\mathcal{D}}(\Omega)=\left\{\boldsymbol{\phi}\in C_c^\infty(\Omega);\ \nabla\cdot\boldsymbol{\phi}=0\right\}.$$

Introducing the norm $\left\|\nabla\boldsymbol{\phi}\right\|_2$ in $\boldsymbol{\mathcal{D}}(\Omega)$, we have the completion $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$.

Exercise 1. • *Show that* $\left\|\nabla\boldsymbol{\phi}\right\|_2$ *is indeed a norm for* $\boldsymbol{\mathcal{D}}(\Omega)$.

- *Prove the following inclusion properties:*

$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)\subset\left\{\boldsymbol{\psi}\in W_0^{1,2}(\Omega);\ \nabla\cdot\boldsymbol{\psi}=0\right\}.$$

Theorem 2. *If* Ω *is Lipschitz, then*

$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)=\left\{\boldsymbol{\psi}\in W_0^{1,2}(\Omega);\ \nabla\cdot\boldsymbol{\psi}=0\right\}.$$

Open Problem 3. *Is it true that*

$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)=\left\{\boldsymbol{\psi}\in W_0^{1,2}(\Omega);\ \nabla\cdot\boldsymbol{\psi}=0\right\}$$

for any bounded Ω*?*

1.3. Weak formulation of the problem

Definition 4. *A measurable vector* $\boldsymbol{v}$ *is said to be a weak solution to* (1) *if*

1. $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$;
2. $\boldsymbol{v}$ *satisfies the following identity:*

$$\nu\left(\nabla\boldsymbol{v},\nabla\boldsymbol{\phi}\right)+\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi}\right)+\left(\boldsymbol{f},\boldsymbol{\phi}\right)=0,\ \forall\ \boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}(\Omega).\tag{2}$$

Once we have shown the existence of $\boldsymbol{v}$, we can recover the pressure p, through the following lemma.

Lemma 5. *Suppose* Ω *is Lipschitz and* $\boldsymbol{F}:W_0^{1,2}(\Omega):\to\boldsymbol{R}$ *is a bounded linear functional such that*

$$\boldsymbol{F}(\boldsymbol{\phi})=0,\ \forall\ \boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega).$$

Then $\exists\, p \in L^2(\Omega)$ *verifies*

$$\boldsymbol{F}(\boldsymbol{\psi}) = (p, \nabla \cdot \boldsymbol{\psi}),\ \forall\, \boldsymbol{\psi} \in W_0^{1,2}(\Omega).$$

Exercise 6. *Prove that we can indeed associate to each weak solution with a pressure* p *using the above lemma.*

1.4. Garlerkin approximated solutions

We establish the existence of weak solutions to (1) by Galerkin method. To this end, let $\{\boldsymbol{\phi}_i\}$ be an orthonormal basis of $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$. We wish to find $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$ such that (2) holds. We look for an "approximate solution":

$$\begin{cases} \boldsymbol{v}_N = c_N^i \boldsymbol{\phi}_i,\ c_N^i \in \boldsymbol{R}, \\ \nu\left(\nabla \boldsymbol{v}_N, \nabla \boldsymbol{\phi}_k\right) + \left(\boldsymbol{v}_N \cdot \nabla \boldsymbol{v}_N, \boldsymbol{\phi}_k\right) + \left(\boldsymbol{f}, \boldsymbol{\phi}_k\right) = 0,\ k = 1, 2, \cdots, N, \end{cases} \tag{3}$$

i.e.

$$\nu c_N^k + c_N^l c_N^s A_{lsk} + F_k = 0,\ \forall\, k = 1, 2, \cdots, N,$$

where

$$\begin{cases} A_{lsk} = \left(\boldsymbol{\phi}_l \cdot \nabla \boldsymbol{\phi}_s, \boldsymbol{\phi}_k\right), \\ F_k = \left(\boldsymbol{f}, \boldsymbol{\phi}_k\right). \end{cases}$$

To show that such $\{c_N^i\}$ exist, we need the following two lemmas.

Lemma 7. *(Brouwer's fixed point theorem) Assume that* $K(\subset \boldsymbol{R}^N)$ *is compact and convex,* $\boldsymbol{P} : K \to K$ *is continuous. Then there exists an* $\boldsymbol{c} \in K$ *such that* $\boldsymbol{P}(\boldsymbol{c}) = \boldsymbol{c}$.

Lemma 8. *Suppose that* $\boldsymbol{P} : \boldsymbol{R}^N \to \boldsymbol{R}^N$ *is continuous and there exists a* $\rho > 0$ *such that*

$$\boldsymbol{P}(\boldsymbol{c}) \cdot \boldsymbol{c} \geq 0,\ \forall\ |\boldsymbol{c}| = \rho. \tag{4}$$

Then $\exists\, c^* \in \bar{B}_\rho$ *such that*

$$\boldsymbol{P}(\boldsymbol{c}^*) = 0.$$

Remark 9. *This is a high-dimensional version of immediate theorem for continuous functions.*

Sketch of Proof of Lemma 8 Argue by contradiction and consider the map

$$B_\rho \ni \boldsymbol{c} \mapsto -\frac{\boldsymbol{P}(\boldsymbol{c})}{|\boldsymbol{P}(\boldsymbol{c})|}\rho \in \partial B_\rho.$$

Using Brouwer's fixed point theorem yields

$$\exists\, \boldsymbol{c}^*,\ s.t.\ -\frac{\boldsymbol{P}(\boldsymbol{c}^*)}{|\boldsymbol{P}(\boldsymbol{c}^*)|}\rho = \boldsymbol{c}^*,$$

thus

$$|\boldsymbol{c}^*| = \rho\ \&\ \boldsymbol{P}(c^*)\cdot c^* = -\frac{|\boldsymbol{c}^*|\cdot|\boldsymbol{P}(\boldsymbol{c}^*)|}{\rho} < 0,$$

contradicting to (4). □

With the lemmas above, one easily verifies

Exercise 10. *The system* (3) *has a solution* $\boldsymbol{v}_N$*.*

Hints Consider

$$\boldsymbol{P}:\boldsymbol{R}^N \ni \left(c_N^i\right)_{i=1}^N \equiv \boldsymbol{c}_N \mapsto \left(\nu\left(\nabla \boldsymbol{v}_N, \nabla \boldsymbol{\phi}_i\right) + \left(\boldsymbol{v}_N\cdot\nabla \boldsymbol{v}_N, \boldsymbol{\phi}_i\right) + \left(\boldsymbol{f}, \boldsymbol{\phi}_i\right)\right)_{i=1}^N \in \boldsymbol{R}^N,$$

and using Poincaré inequality (with best constant denoted by γ_Ω), show that

$$\boldsymbol{P}(\boldsymbol{c}_N)\cdot\boldsymbol{c}_N \geq 0,\ \forall\ |\boldsymbol{c}_N| = \frac{\gamma_\Omega\,\|\boldsymbol{f}\|_2}{\nu}.$$

□

1.5. Passage to limit as $N \to \infty$

Multiplying (3) by c_N^i and sum over i, we easily deduce

$$\|\nabla \boldsymbol{v}_N\|_2 \leq \frac{\gamma_\Omega\,\|\boldsymbol{f}\|_2}{\nu}, \tag{5}$$

where again, γ_Ω is the best constant in Poincar'e's inequality.

The bound (5) implies that

$$\boldsymbol{v}_N \rightharpoonup \boldsymbol{v},\ \text{in}\ \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

and Rellich's compactness theorem yields then

$$\boldsymbol{v}_N \to \boldsymbol{v}, \text{ in } L^2.$$

With these two convergence properties, one easily verifies that

Exercise 11. *we can pass $N \to \infty$ (fix i first) in (3), and lead to*

$$\nu\,(\nabla \boldsymbol{v}, \nabla \boldsymbol{\phi}_i) + (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\phi}_i) + (\boldsymbol{f}, \boldsymbol{\phi}_i) = 0, \ \forall\, i.$$

Also, a density argument shows the existence of a weak solution $\boldsymbol{v}$ to (1).

2. Steady-state Navier-Stokes BVP in an exterior domain

2.1. What is an exterior domain?

Definition 12. $\Omega(\subset \boldsymbol{R}^3)$ *is said to be an exterior domain if it is a domain, complement to a compact set.*

2.2. The problem we are interested in

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{in } \Omega,$$
$$\boldsymbol{v} = 0 \qquad \text{on } \partial\Omega,$$
$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{v}| = 0. \tag{6}$$

2.3. Difficulties we encounter

1. $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$ is no longer compactly imbedded in $L^2(\Omega)$;
2. there is no Poincaré-type inequality for an exterior domain.

Definition 13. *Given $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}_0^{-1,2}(\Omega)$, a measurable vector $\boldsymbol{v}$ is said to be a weak solution to (6) if*

1. $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$;
2. $\nu\,(\nabla\boldsymbol{v}, \nabla\boldsymbol{\phi}) + (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\phi}) + \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle = 0, \ \forall\, \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}(\Omega).$

Remark 14. (6) *is satisfied in a generalized sense:*

$$\lim_{|\boldsymbol{x}|\to\infty}\int_{S^2}|\boldsymbol{v}(|x|,\omega)|\,dS_\omega=0. \tag{7}$$

2.4. Existence of a weak solution

Denote by $\Omega_R=\Omega\cap B_R(0)$, and consider the "approximate problem":

$$\left.\begin{aligned}&\nu\Delta\boldsymbol{v}_n=\boldsymbol{v}_n\cdot\nabla\boldsymbol{v}_n+\nabla p_n+\boldsymbol{f}\\&\nabla\cdot\boldsymbol{v}_n=0\end{aligned}\right\}\ \text{in }\Omega_n,\qquad \boldsymbol{v}_n=0\ \text{on }\partial\Omega_n. \tag{8}$$

The system (8) has at least one weak solution $\boldsymbol{v}_n$ by results in Subsection 1.

By zero-extending, we have $\boldsymbol{v}_n\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$, and the uniform bounds

$$\|\nabla\boldsymbol{v}_n\|_{L^2(\Omega)}\leq\frac{\|\boldsymbol{f}\|_{\boldsymbol{\mathcal{D}}_0^{-1,2}(\Omega)}}{\nu}. \tag{9}$$

Thus

$$\boldsymbol{v}_n\rightharpoonup\boldsymbol{v},\ \text{in }\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

and consequently, one checks that

Exercise 15. *For any $\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}(\Omega)$,*

$$\nu\left(\nabla\boldsymbol{v},\nabla\boldsymbol{\phi}\right)+\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi}\right)+\langle\boldsymbol{f},\boldsymbol{\phi}\rangle=0.$$

Hints

1. By zero-extending, we may always take a linear functional defined on Ω_n as one on Ω.
2. By restriction (to $(supp\ \boldsymbol{\phi})_\varepsilon$—a neighborhood of $supp\ \boldsymbol{\phi}$), we may have strong convergence by invoking Rellich's theorem.

   For example,

$$\left(\boldsymbol{v}_n\cdot\nabla\boldsymbol{v}_n,\boldsymbol{\phi}\right)-\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi}\right)=\left((\boldsymbol{v}_n-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_n,\boldsymbol{\phi}\right)+\left(\boldsymbol{v}\cdot\nabla(\boldsymbol{v}_n-\boldsymbol{v}),\boldsymbol{\phi}\right),$$

the second term being trivially treated, while for the first,

$$\begin{aligned}((\boldsymbol{v}_n-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_n,\boldsymbol{\phi}) &= \sum_{i,j}\int_{(supp\ \boldsymbol{\phi})_\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v})_j\,\partial_j\,(\boldsymbol{v}_n)_i\,\boldsymbol{\phi}_i\\ &= \sum_{i,j}\langle(\boldsymbol{v}_n-\boldsymbol{v}_j)\,\boldsymbol{\phi}_i,\partial_j\,(\boldsymbol{v}_n)_i\rangle_{\left(\boldsymbol{\mathcal{D}}_0^{-1,2}\times\boldsymbol{\mathcal{D}}_0^{1,2}\right)((supp\ \boldsymbol{\phi})_\varepsilon)},\end{aligned}$$

we just need

$$(\boldsymbol{v}_n-\boldsymbol{v}_j)\,\boldsymbol{\phi}_i\to 0, \text{in } \boldsymbol{\mathcal{D}}_0^{-1,2}\left((supp\ \boldsymbol{\phi})_\varepsilon\right),$$

which is easily deduced from the following (compact) imbeddings:

$$\left(\boldsymbol{\mathcal{D}}_0^{1,2}\cap L^6(\Omega)\right)\cdot\boldsymbol{\phi}\subset\boldsymbol{\mathcal{D}}_0^{1,2}\left((supp\ \boldsymbol{\phi})_\varepsilon\right)\subset\subset L^{\frac{6}{5}}\left((supp\ \boldsymbol{\phi})_\varepsilon\right)\subset\boldsymbol{\mathcal{D}}_0^{-1,2}\left((supp\ \boldsymbol{\phi})_\varepsilon\right).$$

3. **Regularity of weak solutions**

We consider the bounded domain case, the exterior domain case being more or less the same.

Before doing this, we need the following

Theorem 16. *(Existence) Suppose Ω is bounded and of class C^{m+2}, $m\geq 0$, $\boldsymbol{f}\in W^{m,q}$, $1<q<\infty$. Then*

$$\exists\,\boldsymbol{u}\in W^{m+2,q}(\Omega),\ \pi\in W^{m+1,q}(\Omega),$$

such that

$$\left.\begin{array}{l}\nu\Delta\boldsymbol{u}=\nabla\pi+\boldsymbol{f}\\ \nabla\cdot\boldsymbol{u}=0\end{array}\right\}\ \text{in } \Omega, \qquad \boldsymbol{u}=0,\quad \text{on } \partial\Omega. \tag{10}$$

Theorem 17. *(Uniqueness) If $\boldsymbol{v}$ is a weak solution to (10), i.e.*

$$\nu\,(\nabla\boldsymbol{v},\nabla\boldsymbol{\phi})+(\boldsymbol{f},\boldsymbol{\phi})=0,\ \forall\,\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

then

$$\boldsymbol{v}=\boldsymbol{u},\ p=\pi+c,\ c\in\boldsymbol{R},$$

where p is the pressure associated to $\boldsymbol{v}$, see Lemma 5.

With the uniqueness theorem above, we have the main result of this subsection.

Theorem 18. *Assume that Ω is a bounded domain in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$. If $\boldsymbol{f} \in C^\infty(\Omega)$, $\Omega \in C^\infty$, then $\boldsymbol{v} \in C^\infty(\Omega)$.*

Proof. The 2-D case being similarly treated, let us concentrated ourselves in the 3-D case.

1.
$$\begin{aligned} &\|\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{f}\|_{3/2} \leq C + \|\boldsymbol{v}\|_6 \|\nabla\boldsymbol{v}\|_2 < \infty \\ \Rightarrow \quad &\boldsymbol{v} \in W^{2,3/2}(\Omega),\ p \in W^{1,3/2}(\Omega); \end{aligned}$$

2. for all $s \in [1, 3/2)$,
$$\left.\begin{aligned} &\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\boldsymbol{f}\|_s \leq \|\boldsymbol{v}\|_{\frac{3s}{3-s}} \|\nabla\boldsymbol{v}\|_3 + C < \infty \\ &\|\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\boldsymbol{f}\|_s \leq \|\nabla\boldsymbol{v}\|_{2s}^2 + \|\boldsymbol{v}\|_{\frac{3s/2}{3/2-s}} \|\nabla^2\boldsymbol{v}\|_{3/2} < \infty \end{aligned}\right\}$$
$$\Rightarrow \quad \boldsymbol{v} \in W^{3,s}(\Omega),\ p \in W^{2,s};$$

3. Iterating as many times as possible, we have finally
$$\boldsymbol{v} \in C^\infty(\Omega),\ p \in C^\infty(\Omega),$$
by invoking Sobolev imbedding, as desired.

□

Remark 19. *The method here is the well-known bootstrap argument.*

Exercise 20. *In 4-D case, prove that one can not show regularity of the weak solutions using the bootstrap argument.*

Remark 21. *We can however, use different method to show smoothness. In a word, we have smooth solutions if $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega) \cap L^n(\Omega)$, and in case $n = 4$,*
$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega) \subset L^4(\Omega).$$

Remark 22. *In case $n = 5$, Struwe has proved some "partial regularity" result.*

Open Problem 23. *Although we can construct smooth solution for $n \geq 5$, can we prove that* **any** *weak solution is regular?*

4. Steady-state body/liquid coupled system

4.1. The problem at hand

Let Ω be an exterior domain in $\boldsymbol{R}^n$. We shall always stay ourselves safely in $\boldsymbol{R}^3$, which is physically relevant.

The PDEs we are concerned is the following

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} - \nabla p = (\boldsymbol{v} - \boldsymbol{\xi} - \boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega, \\ \boldsymbol{v} = \boldsymbol{\xi} + \boldsymbol{\omega}\times\boldsymbol{x} \qquad \text{on } \partial\Omega, \\ \lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{v}(\boldsymbol{x})| = 0; \tag{11}$$

$$\left.\begin{aligned} m\boldsymbol{\omega}\times\boldsymbol{\xi} &= \boldsymbol{G} - \int_{\partial\Omega}\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}d\sigma \\ \boldsymbol{\omega}\times(\boldsymbol{\mathcal{I}}\cdot\boldsymbol{\omega}) &= -\int_{\partial\Omega}\boldsymbol{x}\times\boldsymbol{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}d\sigma \end{aligned}\right\} \tag{12}$$

$$\boldsymbol{\omega}\times\boldsymbol{G} = 0. \tag{13}$$

4.2. Difficulties we encounter

1. The body/liquid system is nonlocally coupled;
2. since the equations are written in a frame attached to the body, the motion of the body is unknown;
3. also, the direction of $\boldsymbol{G} \in S^2$ is not a priori known;
4. S^2 is not a convex set, Brouwer's fixed point theorem could not be applied, we need instead Lefschetz's.

4.3. Function spaces and Kohn's inequality

Denote by

$$\mathcal{R} = \left\{\boldsymbol{\omega} \in C^\infty(\boldsymbol{R}^3);\ \boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{x},\ \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \boldsymbol{R}^3\right\},$$

the rigid motions.

Also, a solenoidal set

$$\boldsymbol{\mathcal{V}}(\Omega) = \left\{ \boldsymbol{\phi} \in C^\infty(\Omega);\ \begin{array}{c} \nabla \cdot \boldsymbol{\phi} = 0 \\ \boldsymbol{\phi} \in \boldsymbol{R} \text{ in a neighborhood of } \partial\Omega \\ \boldsymbol{\phi}(\boldsymbol{x}) = 0 \text{ when } |\boldsymbol{x}| \geq R_{\boldsymbol{\phi}} \end{array} \right\}.$$

Notice that

$$\boldsymbol{\mathcal{D}}(\Omega) \subset \boldsymbol{\mathcal{V}}(\Omega).$$

Introducing a norm

$$\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_2 \equiv \left\| \frac{\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^t}{2} \right\|_2,$$

we have the compeletion $\boldsymbol{V}^{1,2}(\Omega)$.

Exercise 24. *Show that $\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_2$ is indeed a norm for functions $\boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}(\Omega)$.*

Hints One verifies trivially that

$$\begin{aligned} \boldsymbol{\mathcal{D}}\boldsymbol{\phi} = 0 \quad &\Rightarrow \quad \boldsymbol{\phi} \in \boldsymbol{R}(\text{ by simple differential calculus }) \\ &\Rightarrow \quad \boldsymbol{\phi} = 0(\text{ by } \boldsymbol{\phi} = 0 \text{ when } |\boldsymbol{x}| \geq R_{\boldsymbol{\phi}}). \end{aligned}$$

Other hints One can also do this exercise through the following steps:

$$\begin{aligned} \boldsymbol{\mathcal{D}}\boldsymbol{\phi} = 0 \quad &\Rightarrow \quad \Delta\boldsymbol{\phi} = 0 \\ &\Rightarrow \quad \boldsymbol{\phi} = 0(\text{ by } \boldsymbol{\phi} \text{ is analytical }). \end{aligned}$$

Theorem 25. *(Kohn's inequality) If $\boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}(\Omega)$, then*

$$\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_2$$

is equivalent to

$$\|\boldsymbol{\phi}\|_6 + \|\nabla\boldsymbol{\phi}\|_2 .$$

Proof. Extending rigidly outside Ω, we still denote this extension by $\boldsymbol{\phi}$. Then $\boldsymbol{\phi} \in C_0^\infty(\boldsymbol{R}^3)$, and consequently,

$$\begin{aligned}
\|\nabla\boldsymbol{\phi}\|_{2,\Omega}^2 &\leq \|\nabla\boldsymbol{\phi}\|_{2,\boldsymbol{R}^3}^2 \\
&= -\int \Delta\boldsymbol{\phi}\cdot\boldsymbol{\phi}(\text{ integration by parts }) \\
&= -2\int \nabla\cdot\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\cdot\boldsymbol{\phi}(\nabla\cdot\boldsymbol{\phi}=0) \\
&= 2\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_{2,\boldsymbol{R}^3}^2 (\text{ integration by parts }) \\
&= 2\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_{2,\Omega}^2 (\boldsymbol{\phi}(x)\in\boldsymbol{R} \text{ when } \boldsymbol{x}\in\Omega^c),
\end{aligned}$$

and

$$\begin{aligned}
\|\boldsymbol{\phi}\|_{6,\Omega} &\leq \|\boldsymbol{\phi}\|_{6,\boldsymbol{R}^3} \\
&\leq C\|\nabla\boldsymbol{\phi}\|_{2,\boldsymbol{R}^3} \\
&\leq C\|\boldsymbol{\mathcal{D}}\boldsymbol{\phi}\|_{2,\Omega} .
\end{aligned}$$

□

Moreover, we have the following characterization of $\boldsymbol{V}^{1,2}(\Omega)$.

Lemma 26. *Suppose Ω is Lipschitz, then we have*

1. *(equivalent definition)*

$$\boldsymbol{V}^{1,2}(\Omega) = \left\{ \boldsymbol{v}\in W_{loc}^{1,2}(\Omega);\ \begin{array}{c} \nabla\cdot\boldsymbol{v}=0 \\ \|\boldsymbol{v}\|_6 + \|\nabla\boldsymbol{\phi}\|_2 < \infty \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{a}_1 + \boldsymbol{a}_2\times\boldsymbol{x} \text{ for some } \boldsymbol{a}_1,\boldsymbol{a}_2\in\boldsymbol{R}^3 \end{array} \right\};$$

2. *(trace inequality)*

$$|\boldsymbol{a}_1| + |\boldsymbol{a}_2| \leq \kappa\|\boldsymbol{\mathcal{D}}\boldsymbol{v}\|_2 ; \tag{14}$$

3. *(vanishing at infinity)*

$$\lim_{|\boldsymbol{x}|\to\infty} \int_{S^2} |\boldsymbol{v}(|\boldsymbol{x}|, \omega)| \, dS_\omega = 0.$$

4.4. Weak formulation of the problem

Recall that the Cauchy stress tensor

$$\boldsymbol{T}(\boldsymbol{v}, p) = \nu(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^t) - pI,$$

we have

$$\text{LHS of } (11)_1 = \operatorname{div} \boldsymbol{T}(\boldsymbol{v}, p).$$

Hence multiplying $(11)_1$ by $\boldsymbol{\phi} \in \boldsymbol{V}(\Omega)(\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 \times \boldsymbol{x}$ for some $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \boldsymbol{R}^3)$, and from the following computation:

$$\begin{aligned}
\int_\Omega \nabla \cdot \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{\phi} &= \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{\phi} \cdot \boldsymbol{n} d\sigma - \int_\Omega \boldsymbol{\mathcal{D}v} \cdot \boldsymbol{\mathcal{D}\phi} dx \\
&= \boldsymbol{\phi}_1 \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} d\sigma + \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot (\boldsymbol{\phi}_2 \times \boldsymbol{x}) \cdot \boldsymbol{n} d\sigma - (\boldsymbol{\mathcal{D}v}, \boldsymbol{\mathcal{D}\phi}) \\
&= \boldsymbol{\phi}_1 \cdot \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} d\sigma + \boldsymbol{\phi}_2 \cdot \int_{\partial\Omega} x \times \boldsymbol{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} d\sigma - (\boldsymbol{\mathcal{D}v}, \boldsymbol{\mathcal{D}\phi}) \\
&= \boldsymbol{\phi}_1 \cdot \boldsymbol{G}_1 - m\boldsymbol{\phi}_1 \cdot (\boldsymbol{\omega} \times \boldsymbol{\xi}) - \boldsymbol{\omega} \times (\boldsymbol{\mathcal{I}} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{\phi}_2 - (\boldsymbol{\mathcal{D}v}, \boldsymbol{\mathcal{D}\phi}) \text{ (by (12))},
\end{aligned}$$

we have finally

$$\begin{aligned}
&(\boldsymbol{\mathcal{D}v}, \boldsymbol{\mathcal{D}\phi}) + ((\boldsymbol{v} - \boldsymbol{\xi} - \boldsymbol{\omega} \times \boldsymbol{x}) \nabla \boldsymbol{v}, \boldsymbol{\phi}) + (\boldsymbol{\omega} \times \boldsymbol{v}, \boldsymbol{\phi}) \\
&-\boldsymbol{G} \cdot \boldsymbol{\phi}_1 + m\boldsymbol{\omega} \times \boldsymbol{\xi} \cdot \boldsymbol{\phi}_2 + \boldsymbol{\omega} \times (\boldsymbol{\mathcal{I}} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{\phi}_2 = 0,
\end{aligned} \tag{15}$$

for all $\boldsymbol{\phi} \in \boldsymbol{V}(\Omega)$ with $\boldsymbol{\phi}|_{\partial\Omega} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 \times x$.

This naturally leads to the following

Definition 27. *The quadruple* $(\boldsymbol{v}, \boldsymbol{\xi}, \boldsymbol{\omega}, \boldsymbol{G})$ *is said to be a weak solution to* (11)-(13) *if*

1. $\boldsymbol{v} \in \boldsymbol{V}^{1,2}(\Omega)$;
2. $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{\xi} + \boldsymbol{\omega} \times \boldsymbol{x}$;
3. (15) *is satisfied;*
4. $\boldsymbol{\omega} \times \boldsymbol{G} = 0$.

4.5. Existence of a weak solution

As applied twice before, we use Galerkin methold. To this end, we need a special basis.

Exercise 28. $\exists\ \{\boldsymbol{\phi}_k\}_{k=1}^{\infty}$ *such that*

1. $\boldsymbol{\phi}_k \in \boldsymbol{V}(\Omega)$;
2. $(\boldsymbol{\mathcal{D}}\boldsymbol{\phi}_k, \boldsymbol{\mathcal{D}}\boldsymbol{\phi}_{k'}) = \delta_{kk'}$;
3. *the linear hull* $\mathcal{L}$ *of* $\{\boldsymbol{\phi}_k\}$ *in* $\boldsymbol{V}(\Omega)$ *is dense in* $\boldsymbol{V}^{1,2}(\Omega)$;
4. $\forall\ \boldsymbol{\phi} \in \boldsymbol{V}(\Omega),\ \exists\ \{\boldsymbol{\phi}_m\} \subset \mathcal{L}$ *and a compact set* $K \subset \Omega$ *satisfying*
   (a) *supp*$(\boldsymbol{\phi}_m) \subset K$;
   (b) $\lim\limits_{m\to\infty} \|\boldsymbol{\phi}_m - \boldsymbol{\phi}\|_{1,2} = 0$.

Hints $\boldsymbol{V}(\Omega)$ is dense in $\boldsymbol{V}^{1,2}(\Omega)$, which is a separable Hilbert space.

With this basis at hand, we have to construct approximate solution sequence $(\boldsymbol{v}_N, \boldsymbol{\xi}_N, \boldsymbol{\omega}_N, \boldsymbol{G}_N)$ as

$$
\begin{cases}
\boldsymbol{v}_N = c_N^i \boldsymbol{\phi}_i,\ \ \boldsymbol{\phi}_i|_{\partial\Omega} = \boldsymbol{\phi}_{i1} + \boldsymbol{\phi}_{i2} \times \boldsymbol{x} \\
\boldsymbol{\xi}_N = c_N^i \boldsymbol{\phi}_{i1} \\
\boldsymbol{\omega}_N = c_N^i \boldsymbol{\phi}_{i2} \\
\boldsymbol{G}_N \in S^2,
\end{cases}
$$

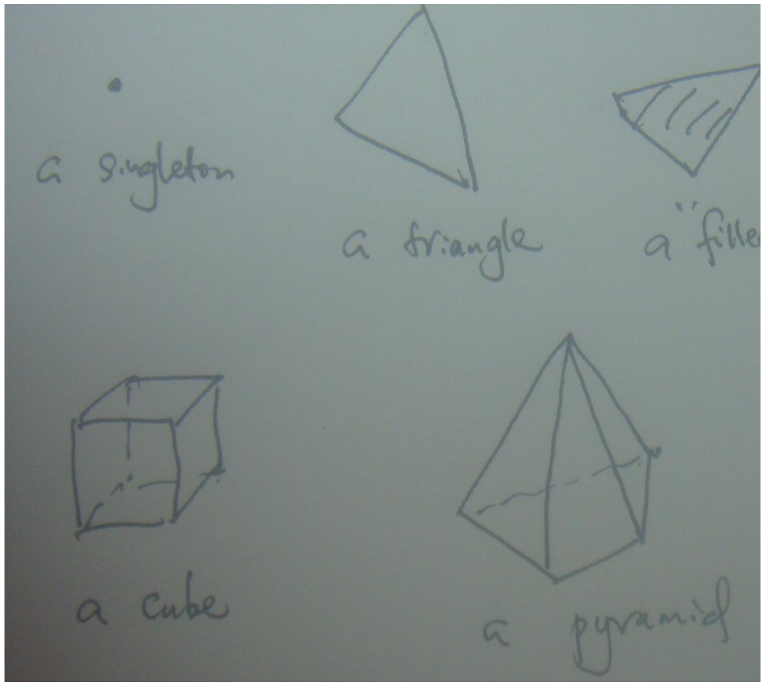
and the following algebraic system is satisfied:

$$
\begin{aligned}
&(\boldsymbol{\mathcal{D}}\boldsymbol{v}_N, \boldsymbol{\mathcal{D}}\boldsymbol{\phi}_k) + ((\boldsymbol{v}_N - \boldsymbol{\xi}_N - \boldsymbol{\omega}_N \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}, \boldsymbol{\phi}_k) + (\boldsymbol{\omega}_N \times \boldsymbol{v}_N, \boldsymbol{\phi}_k) \\
&-\boldsymbol{G}_N \cdot \boldsymbol{\phi}_{1k} + m\boldsymbol{\omega} \times \boldsymbol{\xi}_N \cdot \boldsymbol{\phi}_{2k} + \boldsymbol{\omega}_N \times (\boldsymbol{\mathcal{I}} \cdot \boldsymbol{\omega}_N) \cdot \boldsymbol{\phi}_{2k} \\
&= 0,\ \ k = 1, 2, \cdots, N; \qquad (16) \\
&\boldsymbol{\omega}_N \times \boldsymbol{G}_N = 0.
\end{aligned}
$$

Theorem 29. *The system* (16) *has at least one solution*

$$\boldsymbol{c}_N = \left(c_N^1, \cdots, c_N^N\right), \boldsymbol{G}_N \in S^2.$$

Before going to the proof of Theorem 29, let us review some results in algebraic topology.



Definition 30. *Let P be a polyhedron, the Euler characteristic of which is defined as*

$$\chi(P) = \#(vertices) - \#(edges) + \#(faces).$$

Example 31. *1. If P is a singleton, then*

$$\chi(P) = 1 - 0 + 0 = 1;$$

2. if P is a triangle, then

$$\chi(P) = 3 - 3 + 0 = 0;$$

3. if P is a "filled" triangle, then

$$\chi(P) = 3 - 3 + 1 = 1;$$

4. if P is a cube, then

$$\chi(P) = 8 - 12 + 6 = 2.$$

Exercise 32. *What is $\chi(P)$ if P is pyramid?*

Answer

$$\chi(P) = 5 - 8 + 5 = 2.$$

Definition 33. *Two topological spaces X, Y are said to be homeomorphic if there exists $f : X \to Y$ such that*

1. *f is continuous;*
2. *f is bijective (injective—one-to-one and surjective—onto);*
3. *f^{-1} is continuous.*

Theorem 34. *(Poincaré-Alexander) If P_1 and P_2 are homeomorphic polyhedra (pl. of polyhedron), then*

$$\chi(P_1) = \chi(P_2).$$

Remark 35. *This theorem tells us that Euler characteristic is invariant under homeomorphism.*

With this theorem at our disposal, we may extend the definition of Euler characteristic to more general topological spaces.

Definition 36. *A topological space X is said to be triangulable if it itself homeomorphic to a polyhedron P. In this case, we define*

$$\chi(X) = \chi(P).$$

Remark 37. *Due to Theorem 34, the number $\chi(X)$ is independent of P, which we choose to triangularize X, thus it is an intrinsic number associated to X.*

Example 38. 1. $\chi(S^1) = \chi(\textit{triangle}) = 0$;
2. $\chi(S^2) = \chi(\textit{cube}) = 2$.

Theorem 39. $\chi(\bar{B}^n) = 1, \ \forall\, n \geq 2.$

Theorem 40. *Let X, Y be two compact, triangulable topological spaces, then*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Example 41.

$$\chi(\bar{B}^n \times S^2) = \chi(\bar{B}^n) \times \chi(S^2) = 1 \cdot 2 = 2,\ \forall\, n \geq 2. \tag{17}$$

Definition 42. *Let $X,\ Y$ be two topological spaces, $f,\ g : X \to Y$ is said to be homotopic if there exists a continuous map*

$$H : X \times [0,1] \to Y,$$

such that

1. $H(x,0) = f(x),\ x \in X$;
2. $H(x,1) = g(x),\ x \in X$.

Theorem 43. *(Lefschetz fixed point theorem) Let X be a compact, triangulable topological space, $f : X \to X$ satisfies the following two conditions:*

1. *f is continuous;*
2. *f is homotopic to the identity map.*

Then

$$\chi(X) \neq 0 \Rightarrow f \text{ has a fixed point}.$$

Using this theorem, we have the following

Lemma 44. *Let*

$$\begin{array}{rcl} \boldsymbol{P} &:& \boldsymbol{R}^N \times S^2 \;\to\; \boldsymbol{R}^N \\ \tau &:& \boldsymbol{R}^N \times S^2 \;\to\; TS^2 \end{array}$$

be two Lipschitz continuous maps satisfying the following inequality:

$$\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{e}) \cdot \boldsymbol{\xi} > 0,\ \forall\ |\boldsymbol{\xi}| = \rho.$$

Then $\exists\ (\boldsymbol{\xi}^*,\boldsymbol{e}^*) \in \bar{B}_\rho \times S^2$ such that

$$\boldsymbol{P}(\boldsymbol{\xi}^*,\boldsymbol{e}^*) = 0,\ \tau(\boldsymbol{\xi}^*,\boldsymbol{e}^*) = 0.$$

We postpone the proof of Lemma 44, and deduce easily the

Sketch of Proof of Theorem 29 Consider the map

$$\boldsymbol{P}(\boldsymbol{c}_N, \boldsymbol{G}_N) = \begin{pmatrix} (\boldsymbol{\mathcal{D}}\boldsymbol{v}_N, \boldsymbol{\mathcal{D}}\boldsymbol{\phi}_k) + ((\boldsymbol{v}_N - \boldsymbol{\xi}_N - \boldsymbol{\omega}_N \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}, \boldsymbol{\phi}_k) + (\boldsymbol{\omega}_N \times \boldsymbol{v}_N, \boldsymbol{\phi}_k) \\ -\boldsymbol{G}_N \cdot \boldsymbol{\phi}_{1k} + m\boldsymbol{\omega} \times \boldsymbol{\xi}_N \cdot \boldsymbol{\phi}_{2k} + \boldsymbol{\omega}_N \times (\boldsymbol{\mathcal{I}} \cdot \boldsymbol{\omega}_N) \cdot \boldsymbol{\phi}_{2k} \end{pmatrix}_{k=1}^{N},$$

$$\tau(\boldsymbol{c}_N, \boldsymbol{G}_N) = \boldsymbol{\omega}_N \times \boldsymbol{G}_N,$$

and verify trivially that

1. $\boldsymbol{P}$ and τ are continuous;
2. $\tau(\boldsymbol{c}_N, \boldsymbol{G}_N) \in TS^2$;
3. $\forall\, \rho > \kappa$ (the constant in Lemma 14) > 0,

$$\boldsymbol{P}(\boldsymbol{c}_N, \boldsymbol{G}_N) \cdot \boldsymbol{c}_N > 0, \ \forall\ |\boldsymbol{c}_N| = \rho.$$

The proof of Theorem 29 is then complete by invoking Lemma 44. □

We are now in a position to give the

Proof of Lemma 44 Consider the system of ODEs

$$\begin{cases} \frac{d\boldsymbol{\xi}}{dt} = -\boldsymbol{P}(\boldsymbol{\xi}, \boldsymbol{e}), \\ \frac{d\boldsymbol{e}}{dt} = \tau(\boldsymbol{\xi}, \boldsymbol{e}), \end{cases} \tag{18}$$

where $(\boldsymbol{\xi}(0), \boldsymbol{e}(0)) \in \bar{B}_\rho \times S^2 \equiv X$. Due to the Lipschitz continuity of $\boldsymbol{P}$ and τ, (18) has a C^1 solution on $[0, T)$ for some $T > 0$.

Claim $T = \infty$.

Indeed, we need only show that

$$|\boldsymbol{\xi}(t)| + |\boldsymbol{e}(t)| \leq M, \ \forall\, t \in [0, T), \ (\boldsymbol{\xi}(0), \boldsymbol{e}(0)) \in X,$$

for some $M > 0$ independent of t.

- For $|e(t)|$, since

$$\frac{d\boldsymbol{e}(t)}{dt} = \tau(\boldsymbol{\xi}, \boldsymbol{e}) \in TS^2,$$

we have

$$0 = \frac{d\boldsymbol{e}(t)}{dt} \cdot \boldsymbol{e}(t) = \frac{1}{2}\frac{d}{dt}|\boldsymbol{e}(t)|^2.$$

Thus

$$|\boldsymbol{e}(t)| = |\boldsymbol{e}(0)| = 1.$$

- For $|\boldsymbol{\xi}(t)|$, we claim then

$$|\boldsymbol{\xi}(t)| \leq \rho, \ \forall\, t \in [0, T). \tag{19}$$

(19) is proved by contradiction. That is, suppose (19) is not true, then

$$\exists\, \bar{t} \in [0, T), \ \varepsilon > 0, \ s.t. \ |\boldsymbol{\xi}(\bar{t})| = \rho, \ |\boldsymbol{\xi}(t)| > \rho, \ \forall\, t \in (\bar{t}, \bar{t} + \varepsilon). \tag{20}$$

(This can be done becuase $\boldsymbol{\xi}$ is C^1 in t)

However, (20) can be true since

$$\begin{aligned}
&\frac{d\boldsymbol{\xi}}{dt} = -\boldsymbol{P}(\boldsymbol{\xi}, \boldsymbol{e}) \\
\Rightarrow\ &\frac{1}{2}\frac{d}{dt}|\boldsymbol{\xi}|^2 = -\boldsymbol{P}(\boldsymbol{\xi}, \boldsymbol{e}) \cdot \boldsymbol{\xi} \\
\Rightarrow\ &\frac{1}{2}\frac{d}{dt}|\boldsymbol{\xi}(\bar{t})|^2 = -\boldsymbol{P}(\boldsymbol{\xi}(\bar{t}), \boldsymbol{e}(\bar{t})) \cdot \boldsymbol{\xi}(\bar{t}) < 0 \\
\Rightarrow\ &|\boldsymbol{\xi}(t)| < |\boldsymbol{\xi}(\bar{t})| = \rho \text{ for } t \text{ in a right neighborhood of } \bar{t}.
\end{aligned}$$

The calculations above also show that

$$(\boldsymbol{\xi}(0), \boldsymbol{e}(0)) = (\boldsymbol{\xi}, \boldsymbol{e}) \in X \Rightarrow (\boldsymbol{\xi}(t), \boldsymbol{e}(t)) \in X,$$

i.e. the solution will never leave the manifold X if the initial data are given in X.

Let us now consider a map sequence

$$g_k : X \ni (\boldsymbol{\xi}(0), \boldsymbol{e}(0)) \equiv (\boldsymbol{\xi}, \boldsymbol{e}) \mapsto (\boldsymbol{\xi}(t_k), \boldsymbol{e}(t_k)) \in X,$$

where $t_k = 1/k$.

One verifies trivially that

Exercise 45. *1. g_k is continuous;*

2. g_k is homotopic to the identity map.

Hints The first one being the continuous dependence on the initial data, while for the second one, consider the following ODEs:

$$\begin{cases} \frac{d\boldsymbol{\xi}}{dt} = -s\boldsymbol{P}(\boldsymbol{\xi}, \boldsymbol{e}) \\ \frac{d\boldsymbol{e}}{dt} = s\tau(\boldsymbol{\xi}, \boldsymbol{e}) \end{cases} \tag{21}$$

where $s \in [0, 1]$, $(\boldsymbol{\xi}(0), \boldsymbol{e}(0)) \in X$.

It follows then that (21) has a solution $(\boldsymbol{\xi}(t; s, \boldsymbol{\xi}(0), \boldsymbol{e}(0)), \boldsymbol{e}(t; s, \boldsymbol{\xi}(0), \boldsymbol{e}(0)))$.

We may then define

$$H_k(s, \boldsymbol{\xi}, \boldsymbol{e}) = (\boldsymbol{\xi}(t_k; s, \boldsymbol{\xi}, \boldsymbol{e}), \boldsymbol{e}(t_k; s, \boldsymbol{\xi}, \boldsymbol{e}))\,.$$

Clearly, H_k is continuous and satisfies

$$H_k(0, \boldsymbol{\xi}, \boldsymbol{e}) = (\boldsymbol{\xi}, \boldsymbol{e}),\ H_k(1, \boldsymbol{\xi}, \boldsymbol{e}) = g_k(\boldsymbol{\xi}, \boldsymbol{e}).$$

As (17) shows

$$\chi(X) = \chi(\bar{B}_\rho) \cdot \chi(S^2) = 1 \cdot 2 = 2,\ \forall\, N \geq 2,$$

we may then apply Lifschetz fixed point theorem 43 to deduce

$$\exists\, (\boldsymbol{\xi}_k, \boldsymbol{e}_k) \in X,\ s.t.\ g_k(\boldsymbol{\xi}_k, \boldsymbol{e}_k) = (\boldsymbol{\xi}_k, \boldsymbol{e}_k). \tag{22}$$

Now integrating both sides of (18) from 0 to t_k, we obtain by (22) that

$$\int_0^{t_k} \boldsymbol{P}(\boldsymbol{\xi}(u), \boldsymbol{e}(u))du = 0,\ \int_0^{t_k} \tau(\boldsymbol{\xi}(u), \boldsymbol{e}(u))du = 0.$$

With this information, we may conclude the proof as

- define

  $$F(t) = \int_0^t \boldsymbol{P}(\boldsymbol{\xi}(u), \boldsymbol{e}(u))du,$$

  then

$$\begin{aligned} 0 &= F(0) \\ &= F(t_k) + F'(t_k)(-t_k) + o(t_k) \\ &= -\boldsymbol{P}(\boldsymbol{\xi}(t_k), \boldsymbol{e}(t_k))t_k + o(t_k), \end{aligned}$$

that is,

$$P(\boldsymbol{\xi}(t_k), \boldsymbol{e}(t_k)) = \frac{o(t_k)}{t_k}. \tag{23}$$

Due to the fact that

$$X = \bar{B}_\rho \times S^2$$

is compact, we may assume, up to a subsequence, that

$$(\boldsymbol{\xi}(t_k), \boldsymbol{e}(t_k)) \to (\boldsymbol{\xi}^*, \boldsymbol{e}^*) \in X.$$

Taking $k \to \infty$ in (23), we have

$$\boldsymbol{P}(\boldsymbol{\xi}^*, \boldsymbol{e}^*) = 0.$$

- Similarly, we have

$$\tau(\boldsymbol{\xi}^*, \boldsymbol{e}^*) = 0.$$

The proof of Lemma 44 is now completed. □

Remark 46. *Lemma 44 is for the* $3D$ *case, i.e.* $S^2 \subset \boldsymbol{R}^3$. *More generally, we may consider successfully the odd space dimensions, since*

$$\chi(\bar{B}_\rho \times S^{n-1}) \neq 0,$$

for n *odd.*

Remark 47. *However, when the space dimension* n *is even,*

$$\chi(\bar{B}_\rho \times S^{n-1}) = 0,$$

the Lefschetz fixed point theorm 43 do not apply.

After establishing the existence of a solution to (16) in Theorem 29, we may now do a priori estimates and pass to limit as $N \to \infty$ in (16) for each fixed k.

Exercise 48. *Show that a priori we have*

$$\|\boldsymbol{\mathcal{D}}\boldsymbol{v}_N\|_2 \leq \frac{\kappa}{\nu}, \tag{24}$$

where κ is the constant in (14).

Hints Using the fact

$$((\boldsymbol{a}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v},\boldsymbol{v}) = 0, \quad \text{for a fixed } \boldsymbol{a}\in\boldsymbol{R}^3,$$

we are led to the conservation of energy:

$$\begin{aligned} & \text{dissipation of energy due to viscosity} \\ \equiv\ & \nu\,\|\boldsymbol{\mathcal{D}}\boldsymbol{v}_N\|_2^2 \\ =\ & \boldsymbol{G}_N\cdot\boldsymbol{\xi}_N \\ \equiv\ & \text{work done by gravity.} \end{aligned}$$

from which and (14), (24) is derived.

Now it is the right time to passage to limit.

Exercise 49. *We have naturally the following convergence properties:*

1. $\boldsymbol{\mathcal{D}}\boldsymbol{v}_N \rightharpoonup \boldsymbol{\mathcal{D}}\boldsymbol{v}$ *in* L^2;
2. $\boldsymbol{v}_N \to \boldsymbol{v}$ *in* $L^2(B)$ *for all bounded* $B\subset\Omega$;
3. $\lim_{N\to\infty}\boldsymbol{\xi}_N = \boldsymbol{\xi}$, $\lim_{N\to\infty}\boldsymbol{\omega}_N = \boldsymbol{\omega}$;
4. $\boldsymbol{v} = \boldsymbol{\xi} + \boldsymbol{\omega}\times\boldsymbol{x}$ *on* $\partial\Omega$;
5. $\lim_{N\to\infty}\boldsymbol{G}_N = \boldsymbol{G}$;

for some $(\boldsymbol{v},\boldsymbol{\xi},\boldsymbol{\omega},\boldsymbol{G})$.

Exercise 50. *Using the convergence properties in Exercise 49, and the method in Subsubsection 2.4, show that the quadruple* $(\boldsymbol{v},\boldsymbol{\xi},\boldsymbol{\omega},\boldsymbol{G})$ *is indeed a weak solution to* (11)-(13).

Open Problem 51. *Continuing Remark 47, how can we prove the existence of weak solutions to the steady-state body/liquid coupled system* (11)-(13)?

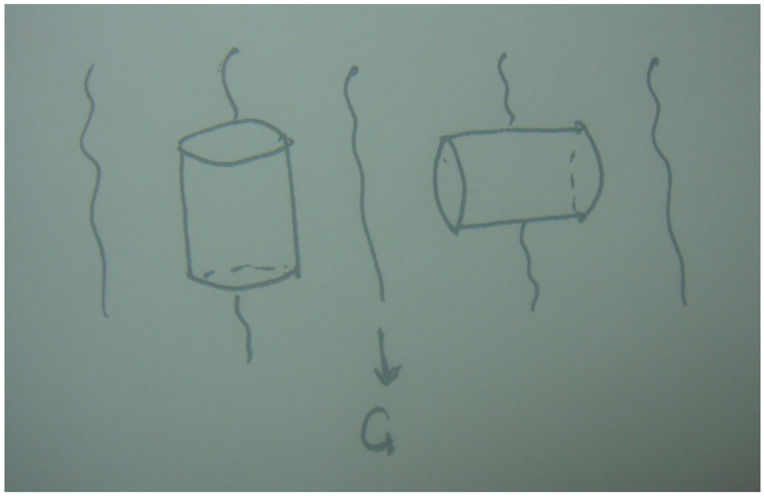


FIGURE 1. Non-uniqueness

4.6. **Non-uniqueness, regularity and related topics**

1. In general, there is no uniqueness. In fact, non-uniqueness is easily proved for certain symmetric bodies. For example, the steady flow as Figure 1 depicted.

   Since non-uniqueness is natural, it would be essential to study the stability properties of the steady-state solutions. Because we observe experimentally only those (steady-state) solutions that are stable, those solutions which are unstable are not so important.

   Unfortunately, the problem of the stability of steady-state solutions is a completely unexplored territory.

2. The regularity of a weak solution is easily done by bootstrap argument as in Subsection 3.

5. Time-dependent falling body problem—a review

We consider the time-dependent counterpart of (11)-(13):

$$\left.\begin{array}{l}\partial_t \boldsymbol{v} + (\boldsymbol{v} - \boldsymbol{\xi} - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = 0 \\ \nabla \cdot \boldsymbol{v} = 0\end{array}\right\} \text{in } \Omega \times (0,T),$$
$$\boldsymbol{v}|_{\partial\Omega \times [0,T)} = \boldsymbol{\xi}(t) + \boldsymbol{\omega}(t) \times \boldsymbol{x},$$
$$m\frac{d\boldsymbol{\xi}}{dt} = \boldsymbol{Q}^t \cdot \boldsymbol{F} - \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{v},p) \cdot \boldsymbol{n} d\sigma - \boldsymbol{\omega} \times \boldsymbol{\xi},$$
$$\boldsymbol{\mathcal{I}} \cdot \frac{d\boldsymbol{\omega}}{dt} = -\int_{\partial\Omega} \boldsymbol{x} \times \boldsymbol{T}(\boldsymbol{v},p) \cdot \boldsymbol{n} d\sigma - \boldsymbol{\omega} \times (\boldsymbol{\mathcal{I}} \cdot \boldsymbol{\omega}), \tag{25}$$
$$\frac{d\boldsymbol{Q}^t}{dt} = \boldsymbol{R}(\boldsymbol{\omega}) \cdot \boldsymbol{Q}^t, \ \boldsymbol{Q}(0) = I, \ \boldsymbol{R}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

Here we use the notation $v(0) = v_0, \ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \ \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$.

The system (25) involves both PDEs and ODEs, and is coupled by a nonlocal coupling.

The weak solution to (25) is trivial, while for the strong solutions, we have

Theorem 52. *(Local existence) Suppose $\Omega(\subset \boldsymbol{R}^3)$ is of class C^2 and $F \in L^2(0,T), \ T > 0$. Assume moreover that $v_0 \in W^{1,2}(\Omega)$ with*

$$\nabla \cdot \boldsymbol{v}_0 = 0, \ \boldsymbol{v}_0|_{\partial\Omega} = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{x}, \ \boldsymbol{\xi}_0, \boldsymbol{\omega}_0 \in \boldsymbol{R}^3.$$

Then $\exists\, T^* \in (0,T)$ and $(\boldsymbol{v}, p, \boldsymbol{\xi}, \boldsymbol{\omega}, \boldsymbol{Q})$ satisfying (25) in $\Omega \times (0,T^*)$. Also,

$$\begin{aligned}&\boldsymbol{v}, \nabla \boldsymbol{v} \in L^\infty(0,T^*; L^2(\Omega)),\\ &\boldsymbol{v} \in L^2(0,T^*; W^{2,2}(\Omega)),\\ &\boldsymbol{\xi}, \boldsymbol{\omega} \in W^{1,2}(0,T^*), \ \boldsymbol{Q} \in W^{2,2}(0,T^*),\\ &\partial_t \boldsymbol{v}, \ \nabla p \in L^2(0,T^*; L^2(\Omega_R)), \ \forall\, R \text{ large}.\end{aligned}$$

Here $\Omega_R = \Omega \cap B_R$.

Theorem 53. *(Global existence) Assume in addition that* $\boldsymbol{F} \in L^2(0,\infty)$. *Then* $\exists\, \delta > 0$ *such that*

$$\|\boldsymbol{v}_0\|_{1,2} + \int_0^\infty |\boldsymbol{F}(t)|^2 \, dt < \delta \Rightarrow T^* = \infty.$$

Theorem 54. *(Asymptotic behavior as* $t \to \infty$*)*

$$\begin{aligned}
&\lim_{t\to\infty} \|\nabla \boldsymbol{v}(t)\|_2 = 0,\\
&\lim_{t\to\infty} |\boldsymbol{\xi}(t)| = 0 = \lim_{t\to\infty} |\boldsymbol{\omega}(t)|\,,\\
&\lim_{t\to\infty} |\boldsymbol{Q}(t) - I| = 0.
\end{aligned}$$

Open Problem 55. *How can we prove the uniqueness of strong solutions to* (25)*?*

6. Steady bifurcation theory of the Navier-Stokes problem in a bounded domain

In this subsection, we consider the system 1.

6.1. What is a bifurcation?

We first state a uniqueness result.

Theorem 56. *(Uniqueness) There exists a* $C = C(\Omega) > 0$ *such that if*

$$R \equiv \frac{\|\boldsymbol{f}\|_2{}^2}{\nu} < C, \tag{26}$$

then the weak solution $\boldsymbol{v}$ *constructed in Subsection 1 is unique.*

Proof. Suppose $\boldsymbol{v}_1$ is another weak solution, then (2) and a simple density argument imply

$$\nu\,(\nabla \boldsymbol{v}, \boldsymbol{\phi}) + (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\phi}) + (\boldsymbol{f}, \boldsymbol{\phi}) = 0,$$

$$\nu\,(\nabla \boldsymbol{v}_1, \boldsymbol{\phi}) + (\boldsymbol{v}_1\cdot\nabla\boldsymbol{v}_1, \boldsymbol{\phi}) + (\boldsymbol{f}, \boldsymbol{\phi}) = 0,$$

for all $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$.

Denote by $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_1$, and taking $\boldsymbol{\phi} = \boldsymbol{u}$, we deduce easily

$$\begin{aligned}
\nu \|\nabla \boldsymbol{u}\|_2^2 &= (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}) + (\boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_1, \boldsymbol{u}) \\
&= -(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}) - (\boldsymbol{v}_1 \cdot \nabla \boldsymbol{u}, \boldsymbol{u}) \\
&= -(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}) \\
&\leq \|\boldsymbol{u}\|_4^2 \|\nabla \boldsymbol{v}\|_2 \\
&\leq C_1 \|\nabla \boldsymbol{u}\|_2^2 \|\nabla \boldsymbol{v}\|_2 ,
\end{aligned}$$

i.e.

$$\|\nabla \boldsymbol{u}\|_2^2 (\nu - C_1 \|\nabla \boldsymbol{v}\|_2) \leq 0. \tag{27}$$

While standard energy estimate shows

$$\begin{aligned}
\nu \|\nabla \boldsymbol{v}\|_2^2 &\leq \|\boldsymbol{f}\|_2 \|\boldsymbol{v}\|_2 \\
&\leq \gamma_\Omega \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2 ,
\end{aligned}$$

that is,

$$\|\nabla \boldsymbol{v}\|_2 \leq \frac{\gamma_\Omega \|\boldsymbol{f}\|_2}{\nu},$$

substituting this into (27), we have

$$\|\nabla \boldsymbol{u}\|_2^2 \left(\nu - \frac{C_1 \gamma_\Omega}{\nu} \|\boldsymbol{f}\|_2\right) \leq 0.$$

Thus $u = 0$ if

$$\frac{\|\boldsymbol{f}\|_2}{\nu^2} < \frac{1}{C_1 \gamma_\Omega} \equiv C.$$

□

Notice that in the above uniqueness result, (26) is important. The natural question is then that what can we say in case (26) is not true, i.e. R is large?

The followings are some questions we are concerned about:

1. When R is large, do uniqueness still hold?

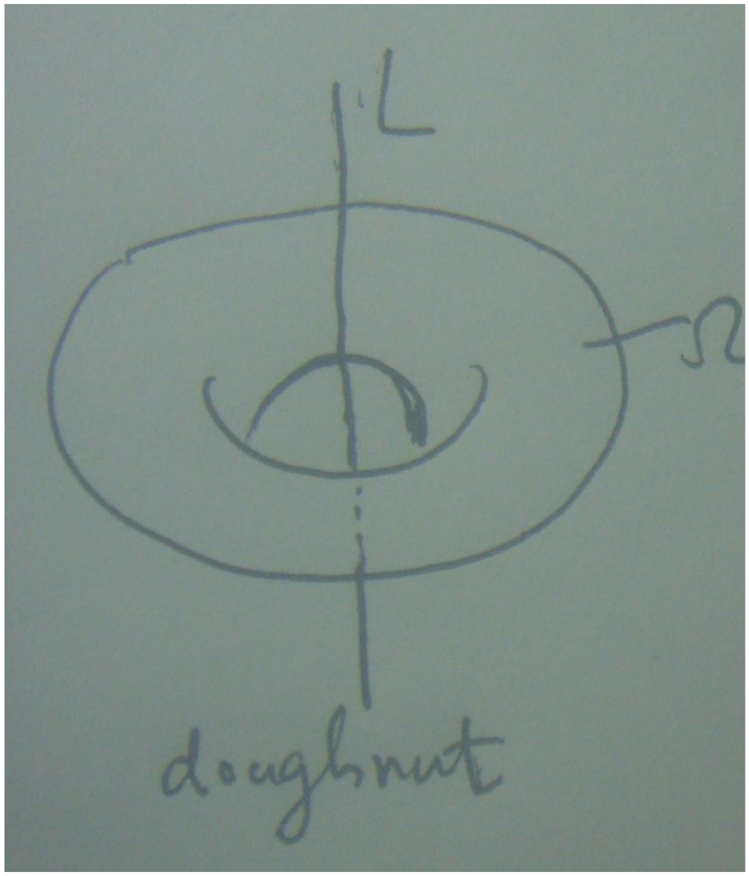


FIGURE 2. Doughnut

2. If it is not so, how many solutions then, and for a fixed $\boldsymbol{f}$, what is the manifold $(\nu, \boldsymbol{v})$, where $\nu \in (0, \infty)$, $\boldsymbol{v}$ is a weak solution to (1)?

For the first question, we give an unaffirmative answer. In particular, we have

Theorem 57. *(Non-uniqueness) Let Ω be a smooth body of revolution around an axis L and Ω does not contain L. For instance, the doughnut as Figure 2 depicted. Then $\exists$ $\boldsymbol{f}$ and $\boldsymbol{v}_*$ such that if $R \geq \gamma_\Omega$, then (1) with boundary data $\boldsymbol{v}_*$ has at least two different solutions.*

The above non-uniqueness result tells us that when R is large, we can not expect uniqueness. This coincides with the observation in experiment. And this is just the starting point of bifurcation.

The second question is then the fundamental issue of bifurcation theory. We give the answer in the next subsubsection.

6.2. Main result—functional/geometrical properties of the set of solutions

We state the main result of this subsection here.

Theorem 58. *1. For any $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}_0^{-1,2}(\Omega)$, system (1) has at least one weak solution;*

2. there exists an open and dense set $\mathcal{R} = \mathcal{R}(\nu)$ such that for each $\boldsymbol{f} \in \mathcal{R}$, system (1) has a finite and odd number $K = K(\nu, \boldsymbol{f})$ of solutions;

3. the number K is constant in each connected component of $\mathcal{R}$.

Remark 59. *The first item in Theorem 58 is done in Subsection 1(with slight modification). But we give a different and unified proof here.*

Proof of Theorem 58

1. Abstract formulation of (1).

   We shall show in Subsubsection 6.3 that the weak solution of (1) can be written as

$$\boldsymbol{M}(\boldsymbol{v}) \equiv \nu \boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) = \boldsymbol{F}, \text{ in } \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

   where $\boldsymbol{N}(\boldsymbol{v})$ and $\boldsymbol{F}$ are suitably defined.

2. Functional properties of $\boldsymbol{M}$.

   We shall show in Subsubsection

   (a) 6.4 that $\boldsymbol{M}$ is proper, i.e. the preimage of a compact set is still compact;

   (b) 6.5 that $\boldsymbol{M}$ is Fredholm of index 0, i.e., the dimension of the kernel of $\boldsymbol{M}$ equals to the codimension of the range of $\boldsymbol{M}$, and both are finite;

   (c) 6.6 that the degree of $\boldsymbol{M}$ is not 0, and conclude the proof of Theorem 58.

□

6.3. **Abstract formulation of (1)**

We shall formulate (1) with $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ as an equation in $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$, where we use the notations in Subsection 1.

Recall that $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$ is a Hilbert space with scalar product

$$[\boldsymbol{v},\boldsymbol{\omega}] = (\nabla\boldsymbol{v},\nabla\boldsymbol{\omega})\,,\ \boldsymbol{v},\ \boldsymbol{\omega}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

and the weak formulation (2) says that

$$\nu\,[\boldsymbol{v},\boldsymbol{\phi}] + (\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi}) + \langle\boldsymbol{f},\boldsymbol{\phi}\rangle\,,\ \forall\,\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega).$$

Observing that

- $|(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi})| \leq \|\boldsymbol{v}\|_4\,\|\nabla\boldsymbol{v}\|_2\,\|\boldsymbol{\phi}\|_4 \leq C\,\|\nabla\boldsymbol{v}\|_2^2\,\|\nabla\boldsymbol{\phi}\|_2\,,$
- $|\langle\boldsymbol{f},\boldsymbol{\phi}\rangle| \leq \|\boldsymbol{f}\|_{\boldsymbol{\mathcal{D}}_0^{-1,2}(\Omega)}\,\|\boldsymbol{\phi}\|_{\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega}\,,$

we have by Riesz representation theorem,

$$\nu\,[\boldsymbol{v},\boldsymbol{\phi}] + [\boldsymbol{N}(\boldsymbol{v}),\boldsymbol{\phi}] = [\boldsymbol{F},\boldsymbol{\phi}]\,,\ \forall\,\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega), \tag{28}$$

where

- $[\boldsymbol{N}(\boldsymbol{v}),\boldsymbol{\phi}] = (\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi})$;
- $[\boldsymbol{F},\boldsymbol{\phi}] = -\langle\boldsymbol{f},\boldsymbol{\phi}\rangle$.

Notice that (28) is equivalent to

$$\boldsymbol{M}(\boldsymbol{v}) \equiv \nu\boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) = \boldsymbol{F},\ \text{in}\ \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega). \tag{29}$$

We shall study various properties of $\boldsymbol{M}$ is the following subsubsections.

6.4. **Properness property of** M

We first establish a sufficient condition to ensure an (nonlinear) operator to be proper, then verify that $\boldsymbol{M}$ satisfies that condition.

Lemma 60. *Let X, Y be Banach spaces, $M:\ X\to Y$ with the following two properties:*

1. $M = H + C$, where H is homeomorphic and C is compact;

2. *M is coercive, i.e.*

$$\|x_n\|_X \to \infty \Rightarrow \|M(x_n)\|_Y \to \infty,\ \forall\ \{x_n\} \subset X. \tag{30}$$

Then M is proper.

Proof. Let $K(\subset Y)$ be compact. For any $\{x_n\} \subset M^{-1}(K)$, we wish to find some convergent subsequence. Since $\{M(x_n)\} \subset K$, we may assume, up to a subsequence(we shall always do so later), that

$$M(x_n) \to y.$$

Coerciveness of M then implies that $\{x_n\}$ is bounded. Invoking the compactness of C, we have

$$C(x_n) \to z.$$

Thus

$$H(x_n) = M(x_n) - C(x_n) \to y - z,$$

and hence

$$x_n \to H^{-1}(y - z).$$

□

Using this lemma, we prove

Lemma 61. *$\boldsymbol{M}$ is proper.*

Proof. Due to the fact

$$\boldsymbol{M}(\boldsymbol{v}) = \nu \boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}),$$

and Lemma 60, we need only show that

1. $\boldsymbol{N}$ is compact.

Let $\{\boldsymbol{v}_n\}$ $(\subset \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega))$ be bounded, we have for $\forall\, \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$,

$$\begin{aligned}[\boldsymbol{N}(\boldsymbol{v}_n) - \boldsymbol{N}(\boldsymbol{v}_m), \boldsymbol{\phi}] &= (\boldsymbol{v}_n \cdot \nabla \boldsymbol{v}_n - \boldsymbol{v}_m \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\phi}) \\ &= (\boldsymbol{v}_n \cdot \nabla (\boldsymbol{v}_n - \boldsymbol{v}_m), \boldsymbol{\phi}) + ((\boldsymbol{v}_n - \boldsymbol{v}_m) \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\phi}) \\ &= -(\boldsymbol{v}_n \cdot \nabla \boldsymbol{\phi}, \boldsymbol{v}_n - \boldsymbol{v}_m) + ((\boldsymbol{v}_n - \boldsymbol{v}_m) \cdot \nabla \boldsymbol{v}_m, \boldsymbol{\phi}),\end{aligned}$$

and thus, up to a subsequence,

$$\begin{aligned}\|\boldsymbol{N}(\boldsymbol{v}_n) - \boldsymbol{N}(\boldsymbol{v}_m)\|_{\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)} &\leq \|\boldsymbol{v}_n\|_4 \|\boldsymbol{v}_n - \boldsymbol{v}_m\|_4 + \|\boldsymbol{v}_n - \boldsymbol{v}_m\|_4 \|\nabla \boldsymbol{v}_m\|_2 \\ &\to 0, \text{ as } n, m \to \infty,\end{aligned}$$

due to the compact imbedding $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega) \subset\subset L^4(\Omega)$.

Now that $\{\boldsymbol{N}(\boldsymbol{v}_n)\}$ is Cauchy in $\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$, we are complete.

2. $\boldsymbol{M}$ is coercive.

We calculate directly as

$$\begin{aligned}[\boldsymbol{M}(\boldsymbol{v}), \boldsymbol{v}] &= \nu [\boldsymbol{v}, \boldsymbol{v}] + [\boldsymbol{N}(\boldsymbol{v}), \boldsymbol{v}] \\ &= \nu \|\nabla \boldsymbol{v}\|_2^2,\end{aligned}$$

thus

$$\nu \|\nabla \boldsymbol{v}\|_2 \leq \|\boldsymbol{M}(\boldsymbol{v})\|_{\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)},$$

and hence

$$\|\boldsymbol{v}_n\|_{\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)} \to \infty \Rightarrow \|\boldsymbol{M}(\boldsymbol{v}_n)\|_{\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)} \to \infty, \ \forall\ \{\boldsymbol{v}_n\} \subset \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega).$$

□

6.5. **Fredholm property of** M

We shall show in this subsubsection that $\boldsymbol{M}$ is Fredholm of index 0.

To this end, we need a series of tools from functional analysis.

Notation 62. • $X,\ Y$ ⇝ *Banach spaces;*

- $\mathcal{L}(X,Y) \leftrightsquigarrow$ *the set of bounded linear operators;*
- $M:\ X\to Y \ \leftrightsquigarrow\ M$ *is a map from* X *to* Y*;*
- $N(M) \leftrightsquigarrow$ *the kernel of* M*;*
- $R(M) \leftrightsquigarrow$ *the range of* M*.*

Definition 63. $L\in\mathcal{L}(X,Y)$ *is said to be Fredholm if*

1. $\alpha = dim\,[N(L)] < \infty$*;*
2. $\beta = codim\,[R(L)] = dim\,[Y/R(M)] < \infty$*.*

And the index of L *is defined as*

$$ind(L) = \alpha - \beta \in \boldsymbol{Z}.$$

Remark 64. *It is well-known that the Fredholm property and the index is invariant under small (w.r.t. norm) or compact perturbations.*

In order to extend the definition of Fredholm property and the index to more general (nonlinear) operators, we need

Definition 65. *A map* $M: X\to Y$ *is said to be (Fréchet) differentiable at* $x_0\in X$*, if there is an* $M'(x_0)\in\mathcal{L}(X,Y)$*, say, such that*

$$\|M(x_0+h)-M(x_0)-M'(x_0)h\|_Y = o\left(\|h\|_X\right),\ \textit{as}\ \|h\|_X\to 0.$$

In this case, $M'(x_0)$ *is said to be the (Fréchet) derivative of* M *at* x_0*.*

Exercise 66. *Show that if* $M'(x_0)$ *exists, then it is unique.*

Exercise 67.

1. *Show that for* $L\in\mathcal{L}(X,Y)$*, we have*

$$L'(x) = L,\ \forall\, x\in X.$$

2. *Show that* $\boldsymbol{N}$ *is (Fréchet) differentiable at any* $\boldsymbol{\omega}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$*, and find the (Fréchet) derivative.*

Remark 68. *For functions defined and with values in* $\boldsymbol{R}$*, the (Fréchet) derivative is the same as the ordinary derivative.*

Remark 69. *One may also consider higher (Fréchet) derivative inductively.*

Theorem 70. *(Inverse function theorem) Let $M : X \to Y$ and $M'(x_0)$ is bijective. Then M is a bijection from a neighborhood of x_0 onto a neighborhood of $M(x_0)$.*

Now we are ready to extend Definition 63 as

Definition 71. *A map $M : X \to Y$ is said to be Fredholm if $M'(x_0)$ is Fredholm for some $x_0 \in X$, and the index of which is defined as*

$$ind(M) = ind\left[M'(x_0)\right].$$

Exercise 72. *Show that the Fredholm property and the index of M is well-defined, i.e., $ind(M)$ is independent of x_0 where we evaluate the (Fréchet) derivative.*

Hints Invoking the invariance of Fredholm property and the index under small perturbations.

We are now ready to state

Lemma 73. *$\boldsymbol{M}$ is Fredholm of index 0.*

Before going to the proof of Lemma 73, we need

Lemma 74. *Suppose that $M : X \to Y$ is compact and (Fréchet) differentiable at $x_0 \in X$. Then $M'(x_0)$ is also compact.*

Proof. Argue by contradiction. Suppose that $M'(x_0)$ is not compact, then $M'(x_0)(\partial B_1)$ does not have compact closure in Y, where ∂B_1 is the unit sphere in X. Thus by Cauchy's convergence criterion,

$$\exists\ \{x_n\} \subset \partial B_1,\ \varepsilon_0 > 0,\ s.t.\ \|M'(x_0)(x_n) - M'(x_0)(x_m)\|_Y \geq \varepsilon_0,\ \forall\, n \neq m.$$

While direct computation shows that

$$\begin{aligned}\|M(x_0+\alpha x_n) - M(x_0+\alpha x_m)\|_Y &\geq |\alpha| \cdot \|M'(x_0)(x_n) - M'(x_0)(x_m)\|_Y \\ &\quad - \|M(x_0+\alpha x_n) - M(x_0) - \alpha M'(x_0)x_n\|_Y\end{aligned}$$

$$- \|M(x_0 + \alpha x_m) - M(x_0) - \alpha M'(x_0)x_n\|_Y$$
$$\geq \ |\alpha| \cdot \varepsilon_0 - o(|\alpha|)$$
$$\geq \ \frac{|\alpha| \cdot \varepsilon_0}{2},$$

if $|\alpha| > 0$ is small and fixed.

Thus $\{M(x_0 + \alpha x_n)\}$ has no convergence subsequence, which contradicts the hypothesis.

□

Now, let us concentrate ourselves in the

Proof of Lemma 73 Recall that

$$\boldsymbol{M} : \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega) \ni \boldsymbol{v} \mapsto \nu \boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega),$$

we have

$$\boldsymbol{M}'(\boldsymbol{v}) = \nu \boldsymbol{I} + \boldsymbol{N}'(\boldsymbol{v}).$$

From the proof in Lemma 61 and Lemma 74, we gather that $N'(v)$ is compact. Thus $\boldsymbol{M}'(\boldsymbol{v})$ is Fredholm of index 0 being the compact perturbation of a homeomorphism.

By definition, we have $\boldsymbol{M}$ is Fredholm of index 0 also. □

6.6. **Degree property of M and the ending of Theorem 58**

In this subsection, we study the degree property of $\boldsymbol{M}$, and conclude the proof of Theorem 58.

As done perviously, we need to recall some basic facts from nonlinear functional analysis.

We use the notations as in Subsubsection 6.5.

Definition 75. *Consider a map $M : X \to Y$. We say that $x \in X$ is a regular point of M if $M'(x)$ is surjective; otherwise, it is a critical point. We also say that $y \in Y$ is a* *regular value if $M^{-1}(y) = \emptyset$ or $M^{-1}(y)$ contains only regular points; otherwise, it is a critical value.*

Notation 76.
- *$\mathcal{R} = \mathcal{R}(M) \leftrightsquigarrow$ the set of regular values of M;*
- *$\mathcal{O} = \mathcal{O}(M) \leftrightsquigarrow$ the set of critical values of M.*

Concerning the set $\mathcal{O}$, we have

Theorem 77. *(Sard's theorem, a simple version)Let $M : \boldsymbol{R}^N \to \boldsymbol{R}^N$, $x_0 \in \boldsymbol{R}^N$ be a critical point of M, i.e.*

$$\det \left[\frac{\partial M_i}{\partial x_j}\right] = 0.$$

Then

$$\mathcal{L}^n(\mathcal{O}) = 0.$$

Theorem 78. *(Smale's theorem) Let X, Y be two separable Banach spaces, $M \in C^k(X,Y)$ be Fredholm of index m. Assume that*

$$k > \max\{m, 0\}.$$

Then $\mathcal{O}$ is of Baire first category, i.e.

$$\mathcal{O} = \cup_{i=1}^\infty E_i,\ E_i^{-o} = \emptyset,$$

and $\mathcal{R}$ is dense in Y.

Moreover, if M is proper, then $\mathcal{R}$ is dense in Y.

Exercise 79. *Let $M \in C^1(X,Y)$ be Fredholm of index 0. Show that for the operator equation*

$$M(x) = y, \tag{31}$$

for a given $y \in Y$,

1. *if $m < 0$, then (31) is not well-posed in the sense of Hadamard; in fact, there is no continuous dependence on y.*

2. *if $m = 0$ and M is proper, then there exists an open and dense set $\mathcal{R}$ in Y such that for all $y \in \mathcal{R}$, (31) has, at most, a finite number of solutions.*

Hints

1. Show that for any $y \in Y$, for which (31) has a solution, we have for any $\varepsilon > 0$, there exists $y_\varepsilon \in \mathcal{R}$ such that
   (a) $M(x) = y_\varepsilon$ has no solution;
   (b) $\|y_\varepsilon - y\| < \varepsilon$.
2. Invoking the inverse function theorem 70 to entail the finiteness.

Before stating another version of Smale's theorem, we introduce

Definition 80. *For a map $M : X \to Y$, the degree of which at $y \in \mathcal{R}$ is defined as*

$$deg(M, y) = \begin{cases} 0, & \text{if } M^{-1}(y) = \emptyset \text{ or } \#M^{-1}(y) \equiv 0 \ (mod\ 2), \\ 1, & \text{if } \#M^{-1}(y) \equiv 1 \ (mod\ 2). \end{cases}$$

Theorem 81. *(Smale-Caccioppoli) Suppose that $M \in C^2(X, Y)$. Then*

$$deg(M, y_1) = deg(M, y_2), \ \forall \, y_1, y_2 \in \mathcal{R}.$$

Due to Theorem 81, we have

Definition 82. *Let $M \in C^2(X, Y)$, we define the degree of M as*

$$deg(M) = deg(M, y), \ y \in \mathcal{R}.$$

Theorem 83. *Let $M \in C^2(X, Y)$ be Fredholm of index 0 and proper. Then the condition*

$$deg(M) \neq 0, \tag{32}$$

implies that

1. *(31) has at least one solution for each $y \in Y$;*
2. *there exists an open and dense set $\mathcal{R} \in Y$ such that (31) has a finite and odd number of solutions.*

Proof. 1. Due to the Smale's theorem 78 and 32, for $\forall\, y \in Y$,

$$\exists\, \mathcal{R} \ni y_n = M(x_n) \to y.$$

Thus $\{y_n\} \cup \{y\}$ is compact. The properness of M then yields some

$$x_{n'} \to x,$$

which then implies

$$M(x_{n'}) \to M(x) = y.$$

2. By Exercise (79), we need only show that $\# M^{-1}(y)$ is odd, while which is easily deduced from (32).

□

Now, we have a sufficient condition to ensure (32), which is easy to check in practice.

Exercise 84. *Let $M \in C^2(X, Y)$ be Fredholm of index 0 and proper. Suppose that*

1. *$\exists\, \bar{y} \in Y$ such that*

$$M(x) = \bar{y}$$

has a unique solution $\bar{x}$;

2. *$M'(\bar{x})$ is injective.*

Then

1. *(31) has at least one solution for each $y \in Y$;*
2. *there exists an open and dense set $\mathcal{R} \in Y$ such that (31) has a finite and odd number of solutions.*

Hints Show trivially using $ind(M) = 0$ that $\bar{x}$ is a regular point of M.

Now we conclude the proof of Theorem 58.

In view of Exercise 79 and Exercise 84, we need only to verify trivially

1. $\boldsymbol{M}(\boldsymbol{v}) \equiv \nu \boldsymbol{v} + \boldsymbol{N}(\boldsymbol{v}) = 0 \Rightarrow \boldsymbol{v} = 0;$
2. $\boldsymbol{M}'(0)(\boldsymbol{\omega}) \equiv \nu \boldsymbol{\omega} = 0 \Rightarrow \boldsymbol{\omega} = 0.$

7. Exam at ∞

1. Let Ω be domain of $\boldsymbol{R}^3$, complement of a compact set B. Consider the following boundary value problem in Ω:

$$\left.\begin{aligned} &\nu\Delta\boldsymbol{v} = (\boldsymbol{v}-\boldsymbol{\xi}-\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}+\nabla p+\boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v}=0 \end{aligned}\right\} \text{ in } \Omega \qquad (33)$$
$$\boldsymbol{v}|_{\partial\Omega}=0,\ \lim_{|x|\to\infty}\boldsymbol{v}(x)=0.$$

We say that $\boldsymbol{v}:\Omega\to\boldsymbol{R}^3$ is a weak solution to (33) if the following conditions are satisfied:

(a) $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)$;

(b) $\boldsymbol{v}$ obeys the following equation:

$$\nu(\nabla\boldsymbol{v},\nabla\boldsymbol{\phi})+((\boldsymbol{v}-\boldsymbol{\xi}-\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v},\boldsymbol{\phi})+(\boldsymbol{f},\boldsymbol{\phi})=0,\ \forall\,\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}(\Omega), \qquad (34)$$

where, we recall,

$$\boldsymbol{\mathcal{D}}(\Omega)\equiv\{\boldsymbol{\phi}\in C_0^\infty(\Omega);\ \nabla\cdot\boldsymbol{\phi}=0\},$$

$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)\equiv\text{ completion of }\boldsymbol{\mathcal{D}}(\Omega)\text{ under the norm }\|\nabla\boldsymbol{\phi}\|_2,$$

and $(\boldsymbol{u},\boldsymbol{\omega})=\displaystyle\int_\Omega\boldsymbol{u}\cdot\boldsymbol{\omega}dx.$

Show that for any $\boldsymbol{\xi},\ \boldsymbol{\omega}\in\boldsymbol{R}^3$ and for every $\boldsymbol{f}$ such that the map

$$\boldsymbol{\mathcal{D}}_0^{1,2}(\Omega)\ni\boldsymbol{\phi}\mapsto(\boldsymbol{f},\boldsymbol{\phi})\in\boldsymbol{R}$$

is a bounded (linear) functional, problem 33 has, at least, on weak solution.

Proof. One easily adapts the details in Subsection 2 and Subsubsection 4.5 to conclude the proof. □

2. Let

$$\mathcal{C}(\Omega)=\left\{\boldsymbol{v}:\Omega\to\boldsymbol{R}^3;\ \begin{array}{l}\boldsymbol{u}\text{ is a weak solution to (33) with }\boldsymbol{\xi}=\boldsymbol{\omega}=0\\ \text{and with given }\boldsymbol{f},\ |\boldsymbol{u}(x)|\le M/|x|,\text{ for some }M>0\end{array}\right\}.$$

Show that there exists a $\gamma > 0$ such that if $\boldsymbol{v} \in \mathcal{C}(\Omega)$ and

$$\sup_{x\in\Omega} \left[(|x|+1)\,|\boldsymbol{v}(x)|\right] < \gamma, \tag{35}$$

then $\boldsymbol{v}$ is the unique weak solution to (33).

Proof. (a) We first establish a Hardy type inequality:

$$\int_\Omega \frac{|\boldsymbol{u}(x)|^2}{(|x|+1)^2} dx \le 4 \int_\Omega |\nabla \boldsymbol{u}(x)|\, dx,\ \boldsymbol{u} \in \boldsymbol{\mathcal{D}}_0^{1,2}(\Omega). \tag{36}$$

Formally,

$$\begin{aligned}
\int_\Omega \frac{|\boldsymbol{u}(x)|^2}{(|x|+1)^2} dx &= \int_\Omega |\boldsymbol{u}(x)|\, d\frac{sgn(x)}{|x|+1} \\
&= -2\int_\Omega u \cdot \nabla \boldsymbol{u} \cdot \frac{sgn(x)}{|x|+1} dx \\
&\le 2\left(\int_\Omega \frac{|\boldsymbol{u}(x)|^2}{(|x|+1)^2} dx\right)^{1/2} \cdot \left(\int_\Omega |\nabla \boldsymbol{u}(x)|\, dx\right)^{1/2},
\end{aligned}$$

and (36) follows readily.

(b) We now prove the problem. Suppose $\boldsymbol{v}_1$ is another weak solution to (33) with $\boldsymbol{\xi} = \boldsymbol{\omega} = 0$. Then (34) and a simple density argument yield

$$\begin{aligned}
\nu \|\nabla \boldsymbol{u}\|_2^2 &= -(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{u}) + (\boldsymbol{v}_1\cdot\nabla\boldsymbol{v}_1, \boldsymbol{u}) \\
&= -(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u}) + (\boldsymbol{v}_1\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) \\
&= -(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{u}) \\
&= (\boldsymbol{u}\cdot\boldsymbol{v}, \boldsymbol{u}) \\
&= \left(\frac{\boldsymbol{u}}{|\cdot|+1}\cdot\nabla\boldsymbol{u}, (|\cdot|+1)\,\boldsymbol{v}\right) \\
&\le \sup_{\boldsymbol{x}\in\Omega}\left[(|x|+1)\,|\boldsymbol{v}(x)|\right]\cdot\left\|\frac{\boldsymbol{u}}{|\cdot|+1}\right\|_2 \|\nabla\boldsymbol{u}\|_2 \\
&< 2\gamma\|\nabla\boldsymbol{u}\|_2^2 \text{ (by (36) and (35))},
\end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_1$. Thus $\boldsymbol{u} = 0$, $\boldsymbol{v} = \boldsymbol{v}_1$, if we take

$$\gamma = \frac{\nu}{2}.$$

□

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