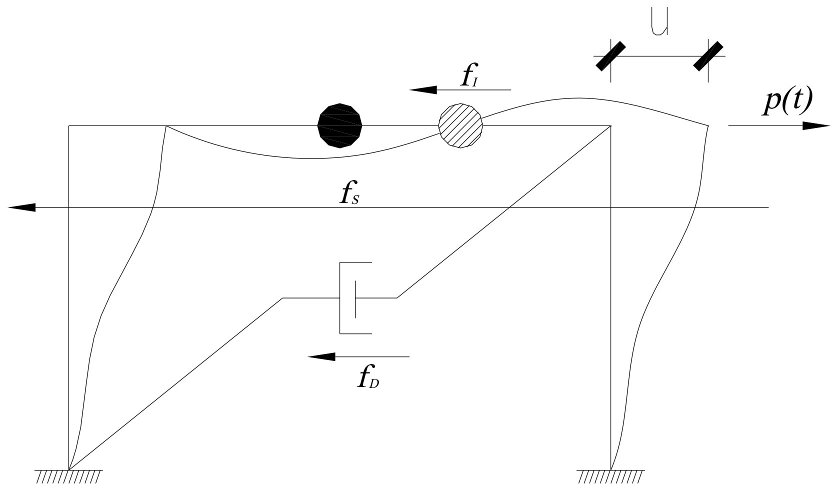
Equations of Motion, and Solution Methods



1 Newton's Second Law of Motion

The external force *p(t)* is taken to be positive in the direction of the x-axis.

The **elastic and damping forces** are shown acting in the opposite direction because they are internal forces that resist the deformation and velocity, respectively.

The resultant force along the x-axis is $p(t) - f_S - f_D$, the Newton's second law of motion gives

$$p(t) - f_S - f_D = m\ddot{u}$$

2 Dynamic Equilibrium

According to **D'Alembert's principle** of dynamic equilibrium, the equation of motion can be written as

$$m\ddot{u} + f_D + f_S = p(t)$$

which means the system is in equilibrium at each time instant.

In this equation, $m\ddot{u}$ represents the term of **inertia force**, f_D represents the term of **damping force**, f_S represents the term of **spring force.**

External forces can be in the form of earthquake excitation exerting to the base of the structure. At each instant of time, the relationship of three displacements, which are ground motion u_g, displacement of mass u^t, the relative displacement between the mass and ground *u*, is

$$u^t(t) = u_g(t) + u(t)$$

The equation of dynamic equilibrium is $f_I + f_D + f_S = 0$, and $f_I = m\ddot{u}^t = m(\ddot{u}_g + \ddot{u})$, which gives

$$m\ddot{u} + f_D + f_S = -m\ddot{u}_g(t)$$

It is assumed that $p(t) = -m\ddot{u}$

3 Solution of the Differential Equation

$$\begin{cases} m\ddot{u} + c\dot{u} + ku = p(t) \\ u\,|_{t=0} = u(0) \quad \dot{u}\,|_{t=0} = \dot{u}(0) \end{cases}$$

3.1 Classical Solution

Complete solution of the linear differential equation of motion consists of the sum of the **complementary solution** $u_c(t)$

and the **particular solution** $u_p(t)$, that is

$$u(t) = u_c(t) + u_p(t)$$

Two **constants of integration** are involved. They appear in the complementary solution and are evaluated from a knowledge of the **initial conditions**.

3.2 Duhamel's Integral

Another well-known approach to the solution of linear differential equations is based on **representing the applied force as a sequence of infinitesimally short impulses.**

The response of the system to an applied force $p(t)$ at time *t* is obtained by adding the response to all impulses up to that time, leading to the following result for an undamped SDF system

$$u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin[\omega_n(t-\tau)]d\tau \ , \ \text{ where } \omega_n = \sqrt{k/m}$$

Duhamel's integral provides an alternative method to the classical solution **if the applied force** $p(t)$ **is defined analytically by a simple function that permits ananytical evaluation of the integral.**

3.3 Frequency-Domain Method

Induce the **Fourier transform**, which leads to the **frequency-domain method** of dynamic analysis

Fourier transform of the excitation function $p(t)$ is given by

$$P(\omega) = \mathcal{F}[p(t)] = \int_{-\infty}^{\infty} p(t)e^{-i\omega t}dt$$

The Fourier transform $U(\omega)$ of the solution $u(t)$ is given by $U(\omega) = H(\omega)P(\omega)$, and the desired solution $u(t)$ is given by the **inverse Fourier transform** of $U(\omega)$:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)P(\omega)e^{i\omega t}d\omega$$

3.4 Numerical Methods

Numerical Methods are practical approach for nonlinear systems and considering the inelastic behavior of structures, including time-stepping methods.