Computational Efficiency of the Finite Element Method based on Second-Order Radiative Transfer Equation

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- **Difficulties in numerical solving RTE**
- **Solution to the difficulties**
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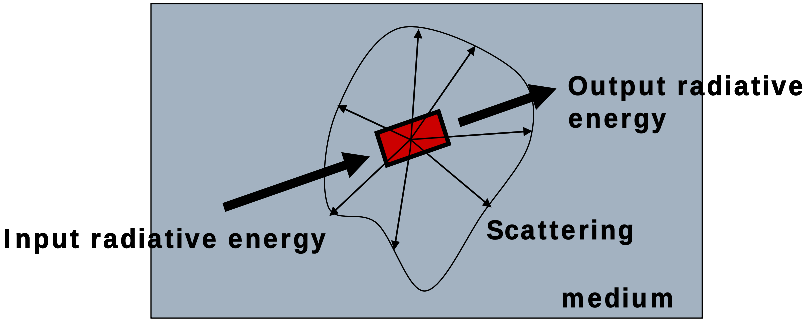
Numerical Insight into the RTE

First-Order Radiative Transfer Equation （FORTE, RTE）：

$$\mathbf{\Omega}\bullet\nabla I+(\kappa_a+\kappa_s)I=\kappa_a I_b+\frac{\kappa_s}{4\pi}\int_{4\pi} I(\mathbf{r},\mathbf{\Omega})\Phi(\mathbf{\Omega},\mathbf{\Omega}')\mathrm{d}\Omega$$



Boundary condition:

$$I(\mathbf{r}_w,\mathbf{\Omega})=\overline{I}_{bw}(\mathbf{r}_w), \qquad \mathbf{n}_w\bullet\mathbf{\Omega}>0$$

Numerical Insight into the RTE

$$\mathbf{\Omega} \bullet \nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int_{4\pi} I(\mathbf{r}, \mathbf{\Omega}') \Phi(\mathbf{\Omega}', \mathbf{\Omega}) \, \mathrm{d}\Omega'$$

- If take direction as "velocity", what it look like?

$$\mathbf{V} \bullet \nabla I + \beta I = S$$

Numerical Insight into the RTE

$$\boldsymbol{\Omega}\bullet\nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi}\int_{4\pi} I(\mathbf{r},\boldsymbol{\Omega}')\,\Phi(\boldsymbol{\Omega}',\boldsymbol{\Omega})\,\mathrm{d}\Omega'$$

- If take direction as "velocity", what it look like?

$$\mathbf{V}\bullet\nabla I + \alpha\nabla^2 I + \beta I = S$$

Numerical Insight into the RTE

$$\boldsymbol{\Omega}\bullet\nabla I + \beta I = \kappa_a I_b + \frac{\kappa_s}{4\pi}\int_{4\pi} I(\mathbf{r},\boldsymbol{\Omega}')\Phi(\boldsymbol{\Omega}',\boldsymbol{\Omega})\,\mathrm{d}\Omega'$$

- If take direction as "velocity", what it look like?

$$\mathbf{V}\bullet\nabla I + \alpha\nabla^2 I + \beta I = S$$

- ***Due to the absence of diffusion term, it is a Convection-dominated type of general convection diffusion equation***

Difficulties in Numerical Solving RTE

1) *Stability problem*

- Caused by ***Convection-dominated*** characteristics of RTE

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- Num. Phenomena: False energy diffusion
- Cause: Insufficient spatial accuracy

Difficulties in Numerical Solving RTE

1) *Stability problem*

- Caused by ***Convection-dominated*** **characteristics** of RTE

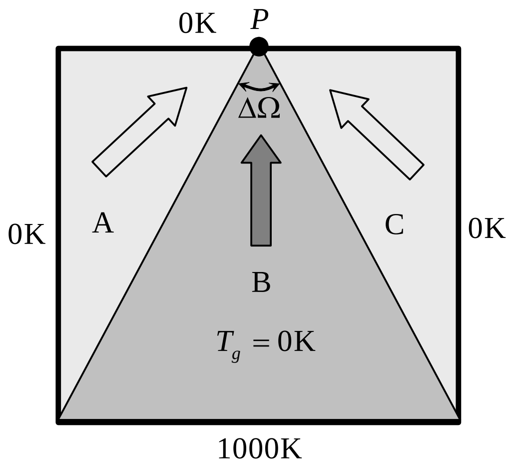
2) *False Scattering*

- Num. Phenomena: False energy scattering
- Cause: Insufficient spatial accuracy

3) "Ray Effects"

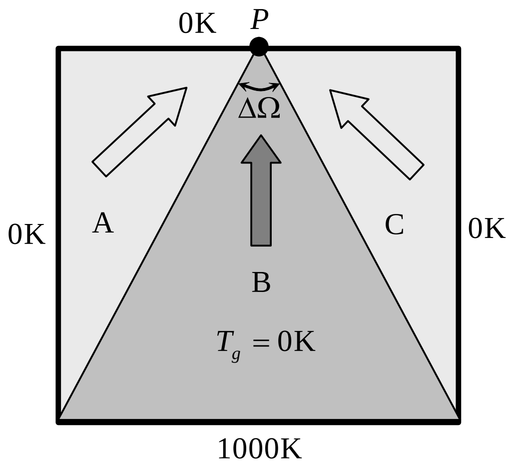
- Num. Phenomena: nonphysical wiggles in results
- Cause: Insufficient angular quadrature accuracy, **may coupled with 1) and 2)**

Example of Causes of *Ray effects*

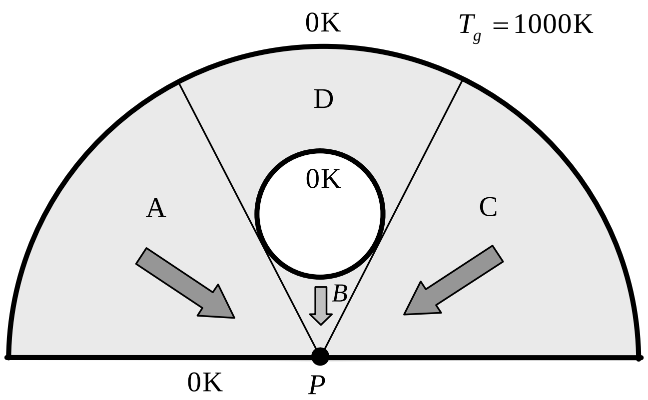


Boundary load with large nonuniformity

Example of Causes of *Ray effects*

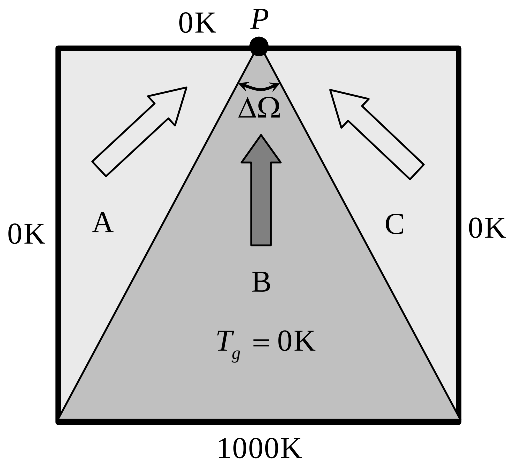


Boundary load with large nonuniformity

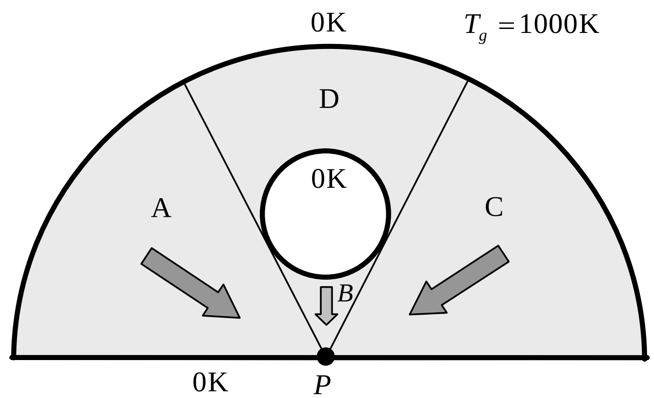


Interior Obstacle shielding

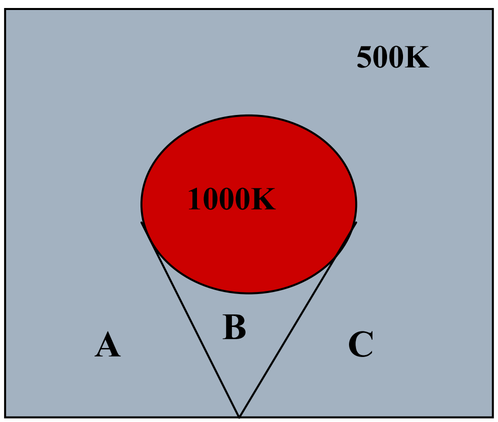
Example of Causes of *Ray effects*



Boundary load with large nonuniformity



Interior Obstacle shielding



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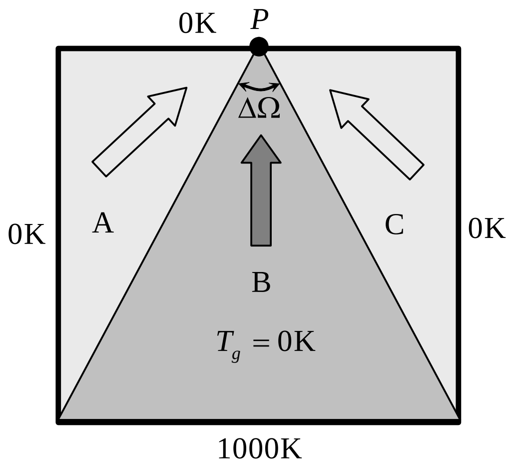
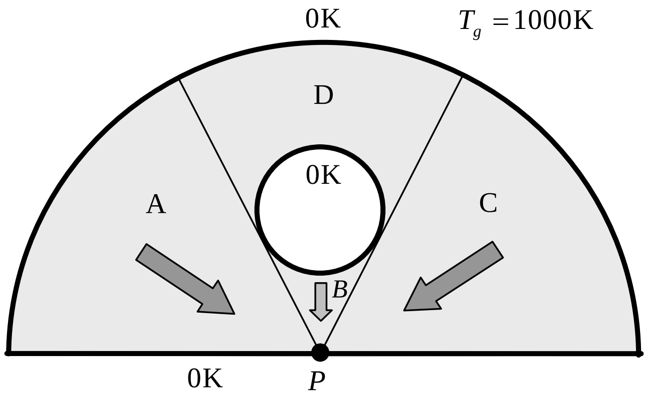
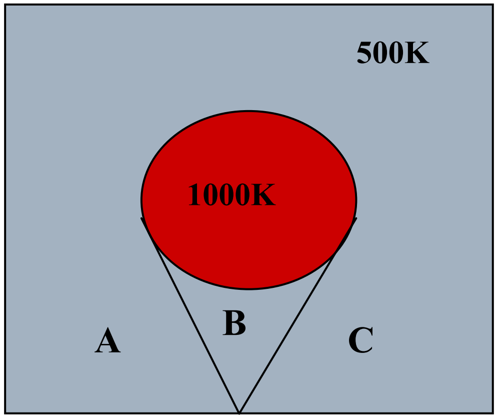
Example of Causes of *Ray effects*



Boundary load with large nonuniformity



Interior Obstacle shielding



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Zhao & Liu (*JQSRT*, 2007)

Solution to Difficulties

1) *Stability problem*

- Basic rationale
  - (A) Upwinding in discretization, artificial diffusion …
  - (B) Transform FORTE to second order equation, or cancel the convection term
    - Even-Parity formulation of RTE (EPF-RTE)
    - Second Order RTE (SORTE)

2) *False Scattering*

3) "Ray Effects"

Advantage of Solution (B)

- No artificial diffusion is needed to be intentionally added
- Based on the equation, radiative transfer can be solved stably with many methods, FEM, FVM, Meshless method,
- Hence it is a **unified approach,** one for all

Disadvantage of EPF-RTE

- Solution variable is not radiative intensity
- Difficult to be extended to anisotropic scattering media

Introduction to SORTE

Derivation

$$\frac{\mathrm{d}}{\mathrm{d}s} I + \beta I = \underbrace{\kappa_a I_b + \frac{\kappa_s}{4\pi} \int_{4\pi} I(\mathbf{r}, \mathbf{\Omega}') \Phi(\mathbf{\Omega}', \mathbf{\Omega}) \, \mathrm{d}\Omega'}_{S}$$

Introduction to SORTE

Derivation

$$\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}I + I = \frac{1}{\beta}S$$

Introduction to SORTE

Derivation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}I + I = \frac{1}{\beta}S\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}I\right] + \frac{\mathrm{d}}{\mathrm{d}s}I = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\beta}S\right]$$

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}I\right] + \beta I = S - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\beta}S\right]$$

Introduction to SORTE

Second Order Radiative Transfer Equation (SORTE) [*Zhao & Liu(2007)*]:

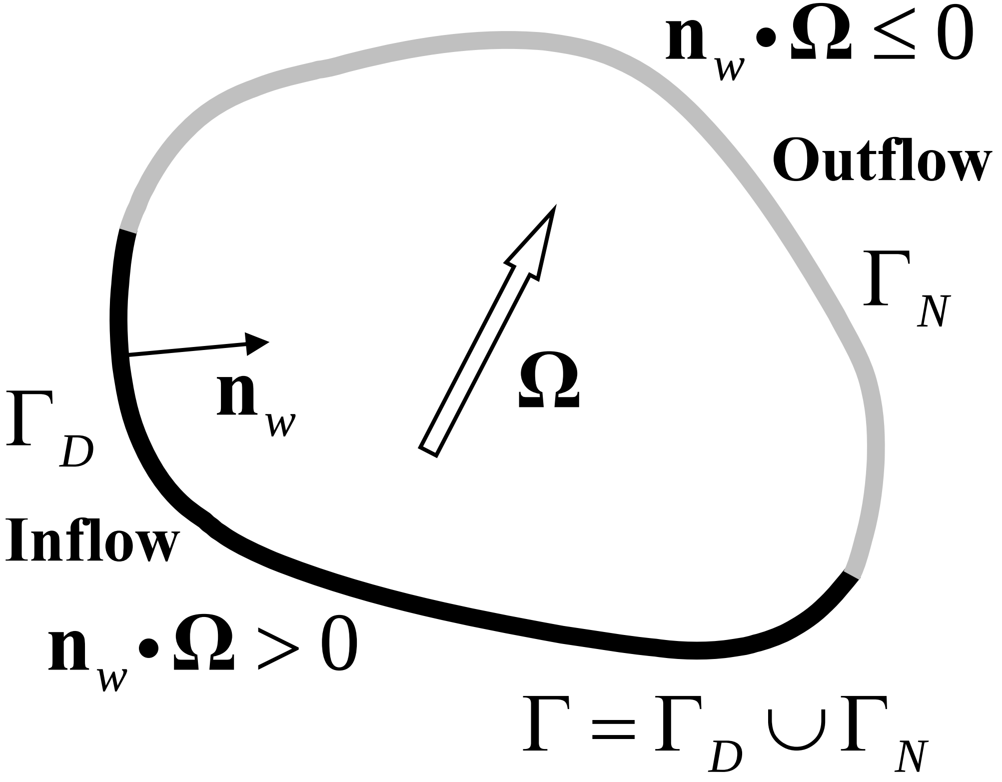
$$-\beta^{-1}\mathbf{\Omega}\bullet\nabla\left[\beta^{-1}\mathbf{\Omega}\bullet\nabla I\right]+I=S-\beta^{-1}\mathbf{\Omega}\bullet\nabla S$$

Properties of the SORTE

- Convection term is cancelled and replaced by a diffusion term
- Solution variable is intensity
- Can be easily applied to anisotropic scattering media, without limit on general applicability of FORTE

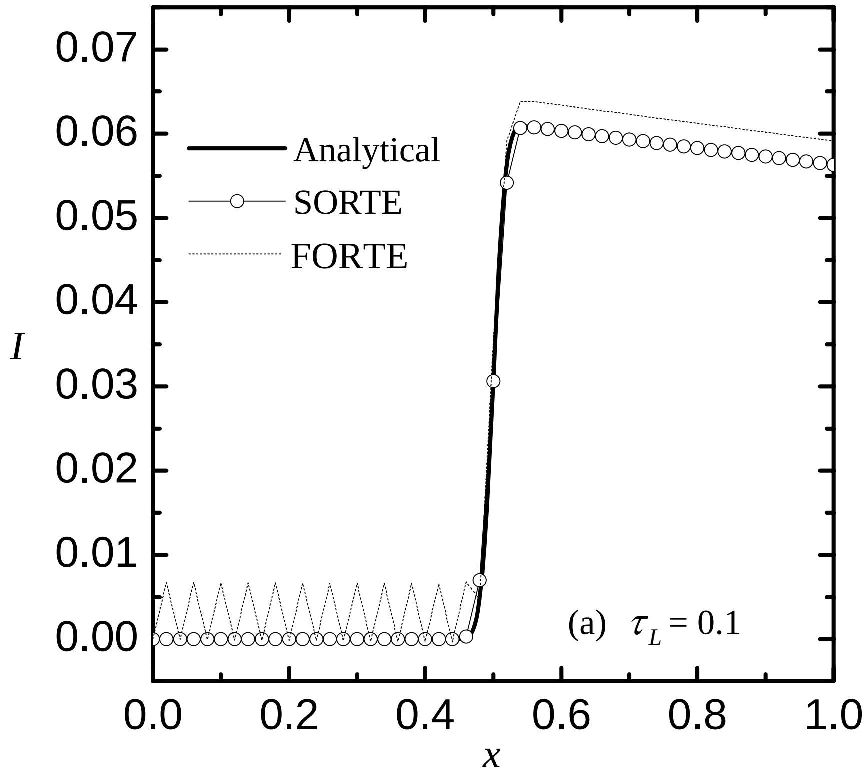
Introduction to SORTE

Boundary conditions



SORTE Performance Demo.

Solved intensity distribution in a slab with a Gaussian hill source



[Zhao & Liu(2007)]

Then, However,

- ***What is the weakness of the SORTE?***
  - *Probably the most important is the computational efficiency of the numerical methods based on it*
- ***Which is crucial for broad application of this approach.***
- ***As such, this subject forms the major motivation of present research***

Objectives of this work

Investigate the

accuracy and solution cost of finite element method (FEM) based on the SORTE

Formulation and Implementation

SORTE in 2D can be written as:

$$(\mu^m)^2 \frac{\partial^2 I^m}{\partial x^2} + (\eta^m)^2 \frac{\partial^2 I^m}{\partial y^2} + 2\mu^m \eta^m \frac{\partial^2 I^m}{\partial x \partial y} - \beta^2 I^m = U^m$$

$$U^m = \mathbf{\Omega}^m \bullet \nabla S^m - \beta S^m$$

Formulation and Implementation

FEM discretization of the SORTE:

- 1) *FEM* approximation

$$I^m(\mathbf{r}) \simeq \sum_{i=1}^{N_{sol}} I_i^m \phi_i(\mathbf{r})$$

- 2) *Galerkin* approach

$$\sum_{i=1}^{N_{sol}} I_i^m \int_V \left[(\mu^m)^2 \frac{\partial^2 I^m}{\partial x^2} + (\eta^m)^2 \frac{\partial^2 I^m}{\partial y^2} + 2\mu^m \eta^m \frac{\partial^2 I^m}{\partial x \partial y} - \beta^2 I^m \right] \phi_j(\mathbf{r}) \mathrm{d}V$$

$$= \int_V U^m(\mathbf{r})\, \phi_j(\mathbf{r}) \mathrm{d}V$$

- 3) final matrix form: $\mathbf{K}^m \mathbf{I}^m = \mathbf{H}^m$

Formulation and Implementation

By using tool matrices approach [Zhao & Liu (2006)]:

$$\mathbf{K}^m = (\mu^m)^2\mathbf{A}^{xx} + \mu^m\eta^m\mathbf{A}^{xy} + \eta^m\mu^m(\mathbf{A}^{xy})^T + (\eta^m)^2\mathbf{A}^{yy} + \beta^2\mathbf{B}^{oo} + \beta\left(\mu^m\mathbf{N}^x + \eta^m\mathbf{N}^y\right)$$

$$\mathbf{H}^m = \left[\beta\mathbf{B}^{oo} - \mu^m(\mathbf{B}^{xo})^T - \eta^m(\mathbf{B}^{yo})^T + \beta\left(\mu^m\mathbf{N}^x + \eta^m\mathbf{N}^y\right)\right]\mathbf{S}^m$$

$$A_{ji}^{xx} = \int_V \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\mathrm{d}V \quad A_{ji}^{xy} = \int_V \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial y}\mathrm{d}V \quad A_{jn}^{yy} = \int_V \frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\mathrm{d}V$$

$$B_{ji}^{xo} = \int_V \frac{\partial\phi_j}{\partial x}\phi_i\,\mathrm{d}V \quad B_{jn}^{yo} = \int_V \frac{\partial\phi_j}{\partial y}\phi_i\,\mathrm{d}V \quad B_{jn}^{oo} = \int_V \phi_j\phi_i\,\mathrm{d}V$$

$$N_{ji}^{x} = \int_{\Gamma_N} \phi_j\phi_i\,(\mathbf{n}_w\bullet\mathbf{i})\mathrm{d}A \quad N_{ji}^{y} = \int_{\Gamma_N} \phi_j\phi_i\,(\mathbf{n}_w\bullet\mathbf{j})\mathrm{d}A$$

Formulation and Implementation

FEM discretization of the FORTE:

- 1) *FEM* approximation

$$I^m(\mathbf{r}) \simeq \sum_{i=1}^{N_{sol}} I_i^m \phi_i(\mathbf{r})$$

- 2) *Galerkin* and *Least Squares* approach
- 3) final matrix form:

$$\mathbf{K}^m \mathbf{I}^m = \mathbf{H}^m$$

Formulation and Implementation

FORTE with Galerkin scheme:

$$\mathbf{K}^{m} = \mu^{m}(\mathbf{B}^{xo})^{T} + \eta^{m}(\mathbf{B}^{yo})^{T} + \beta\mathbf{B}^{oo}$$

$$\mathbf{H}^{m} = \mathbf{B}^{oo}\mathbf{S}^{m}$$

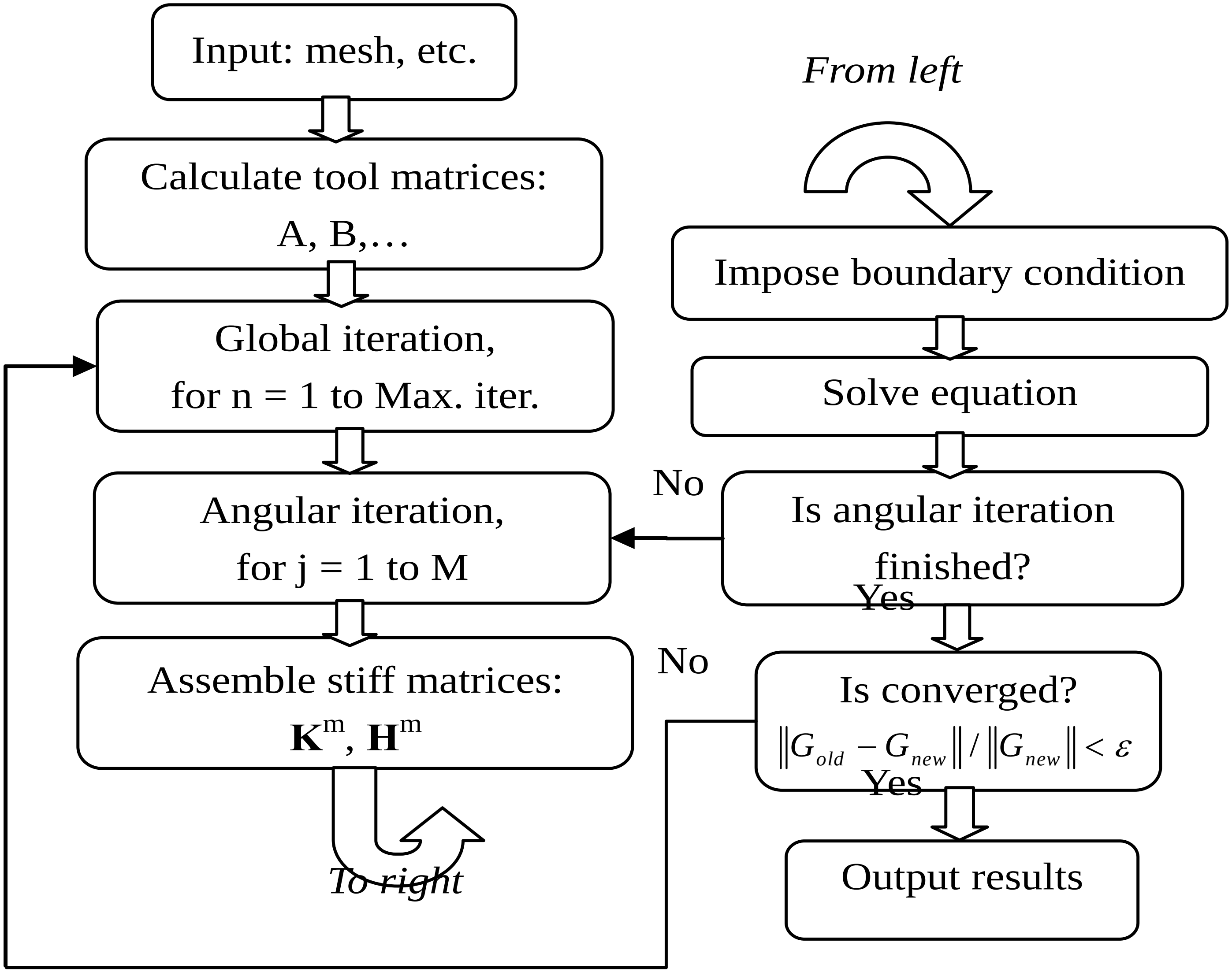
Formulation and Implementation

FORTE with Least square scheme:

$$\mathbf{K}^m = (\mu^m)^2 \mathbf{A}^{xx} + \mu^m \eta^m \mathbf{A}^{xy} + \mu^m \beta \mathbf{B}^{xo}$$
$$+\eta^m \mu^m (\mathbf{A}^{xy})^T + (\eta^m)^2 \mathbf{A}^{yy} + \eta^m \beta \mathbf{B}^{yo}$$
$$+\beta \mu^m (\mathbf{B}^{xo})^T + \beta \eta^m (\mathbf{B}^{yo})^T + (\beta)^2 \mathbf{B}^{oo}$$

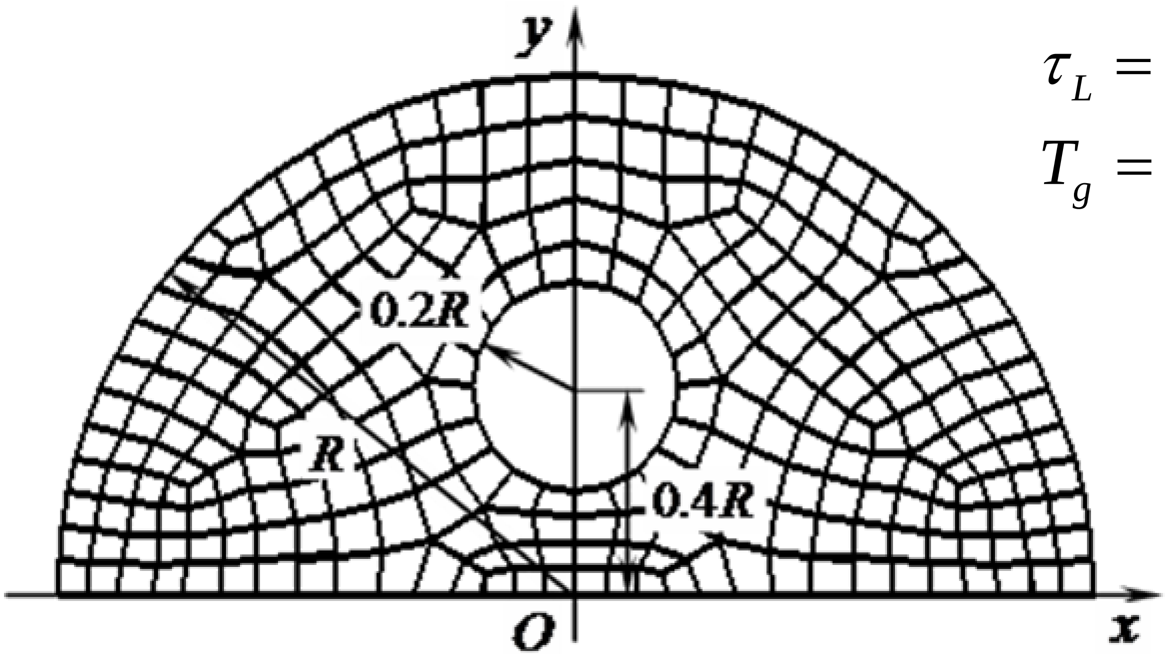
$$\mathbf{H}^m = \left(\mu^m \mathbf{B}^{xo} + \eta^m \mathbf{B}^{yo} + \xi^m \mathbf{B}^{zo} + \beta \mathbf{B}^{oo} \right) \mathbf{S}^m$$

Generic Solution procedures



Results and Discussion

Case 1: Semicircular enclosure with a circular hole

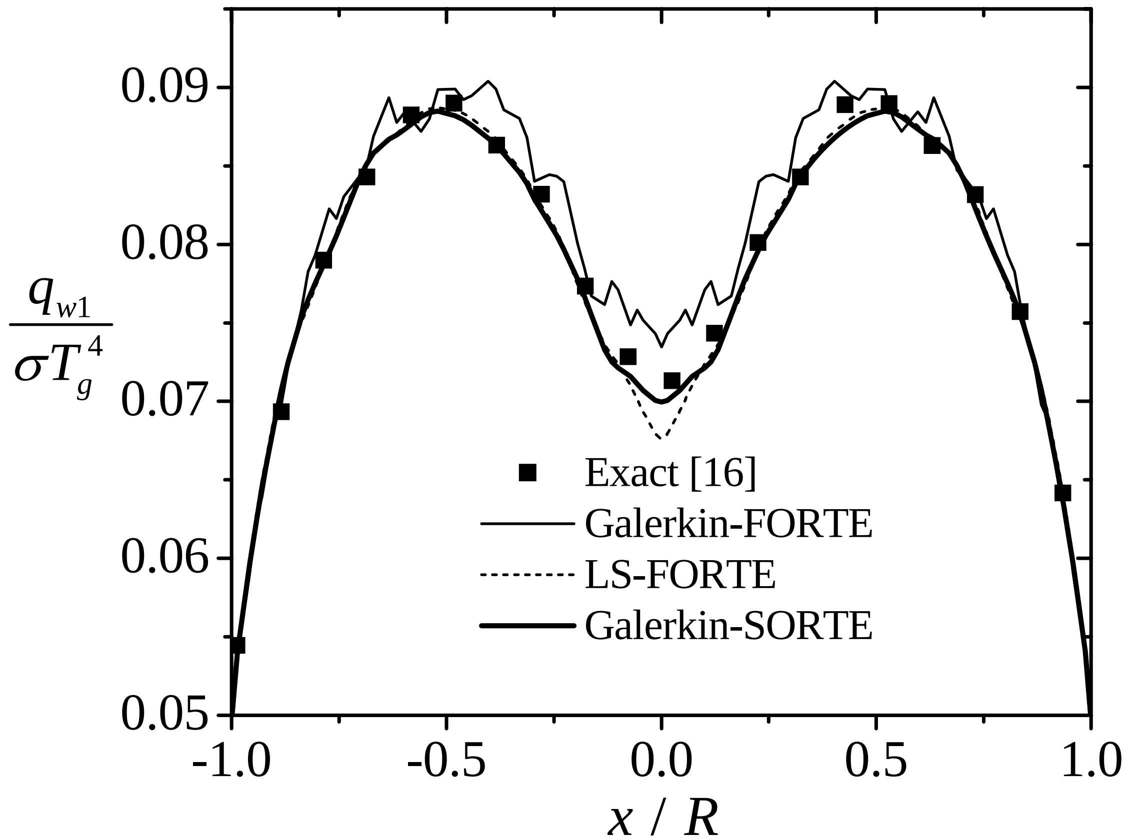


$$\tau_L = \beta R = 0.1$$

$$T_g = 1000\ \mathrm{K}$$

Configuration of the semicircular enclosure and mesh decomposition (272 elements).

Results and Discussion



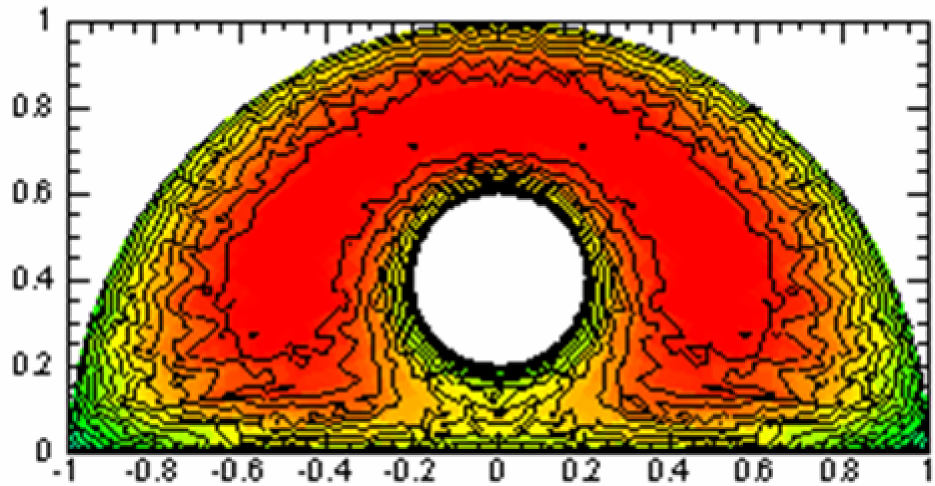
Heat flux distribution along bottom wall

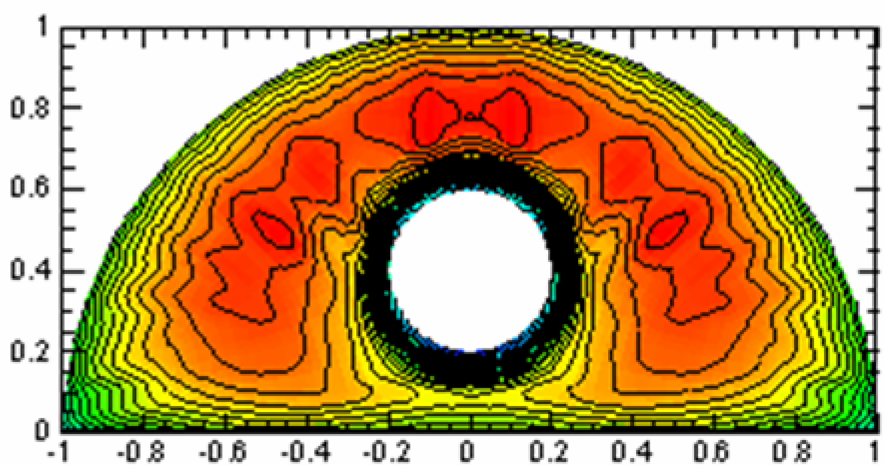
Space: 272 elements, shape function is constructed through 3rd order Chebyshev approximation,

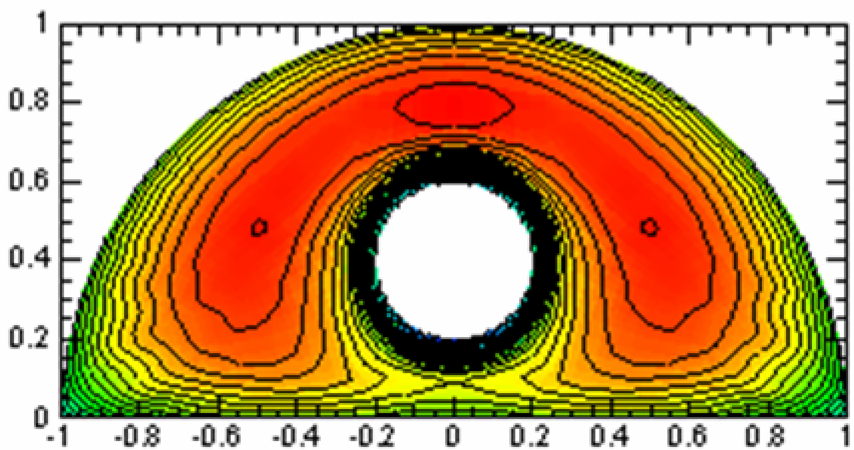
Solid angle: $N_\theta \times N_\phi = 20 \times 40$

Results and Discussion

Galerkin-FORTE





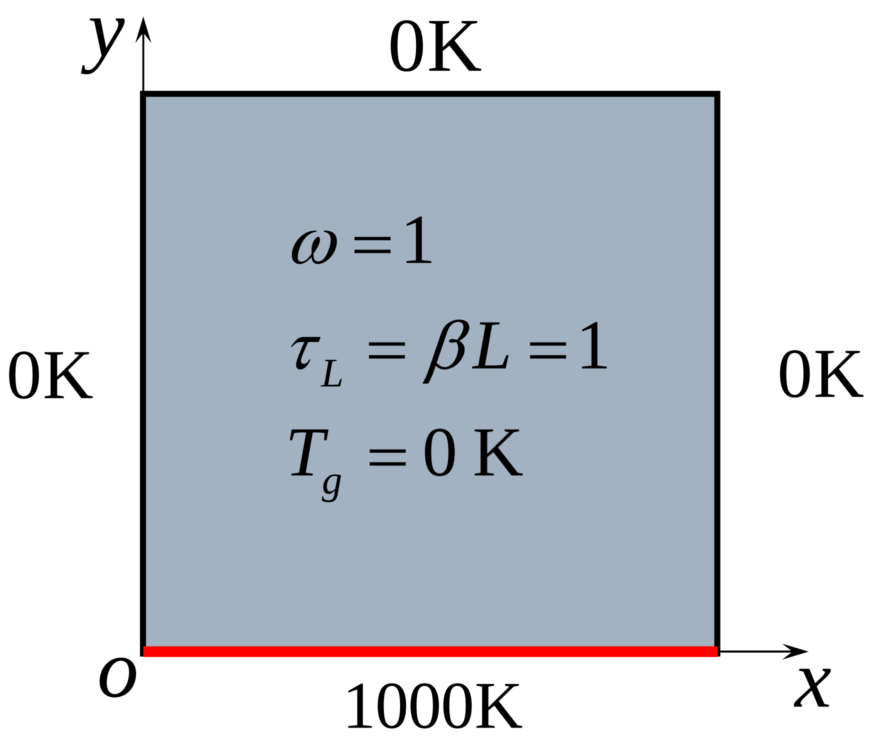


LS-FORTE

Galerkin-SORTE

Results and Discussion

Case 2: Isotropically Scattering Medium in a Square Enclosure

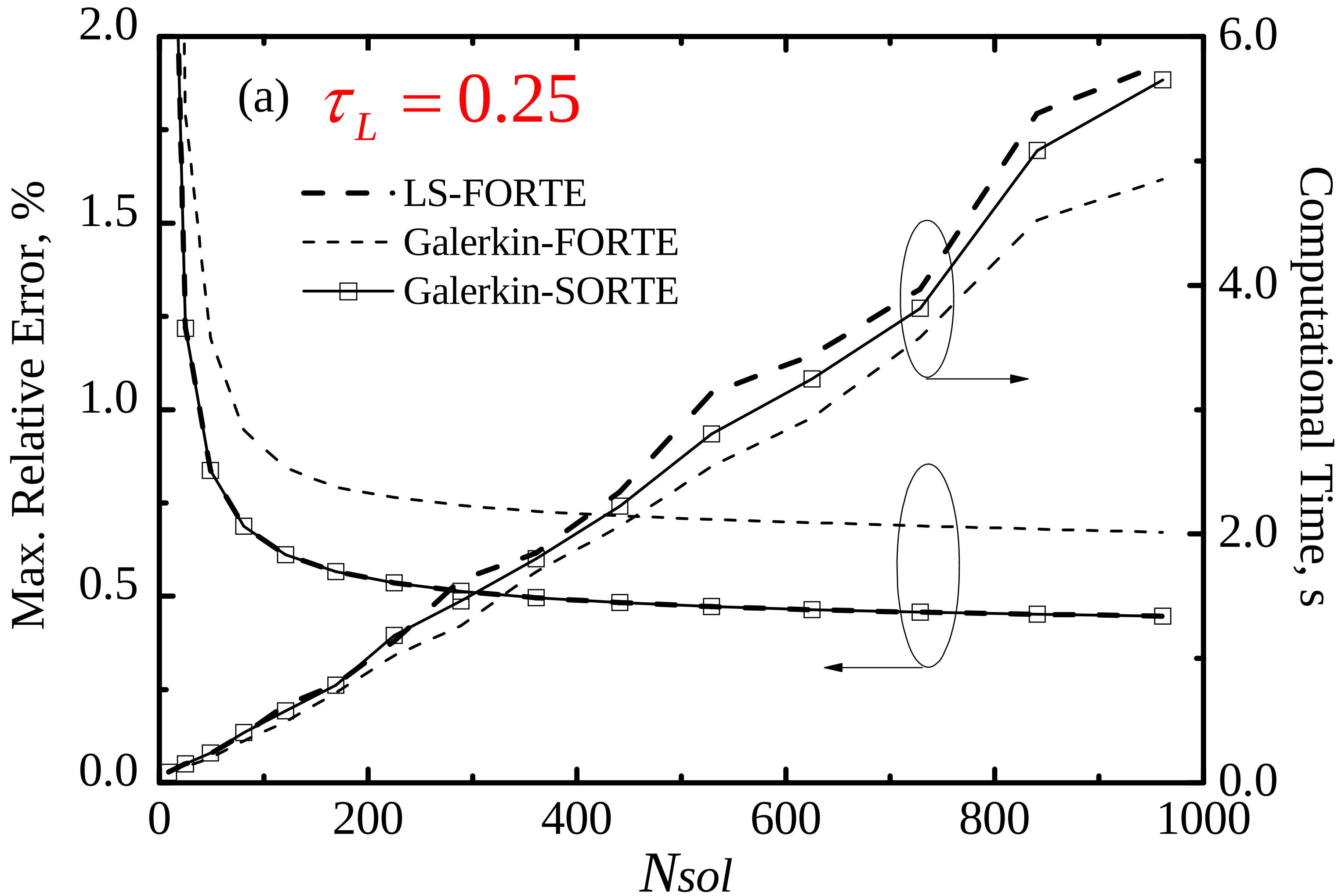


Space: $M \times M$ *bilinear* elements, M is taken as needed for convergence analysis

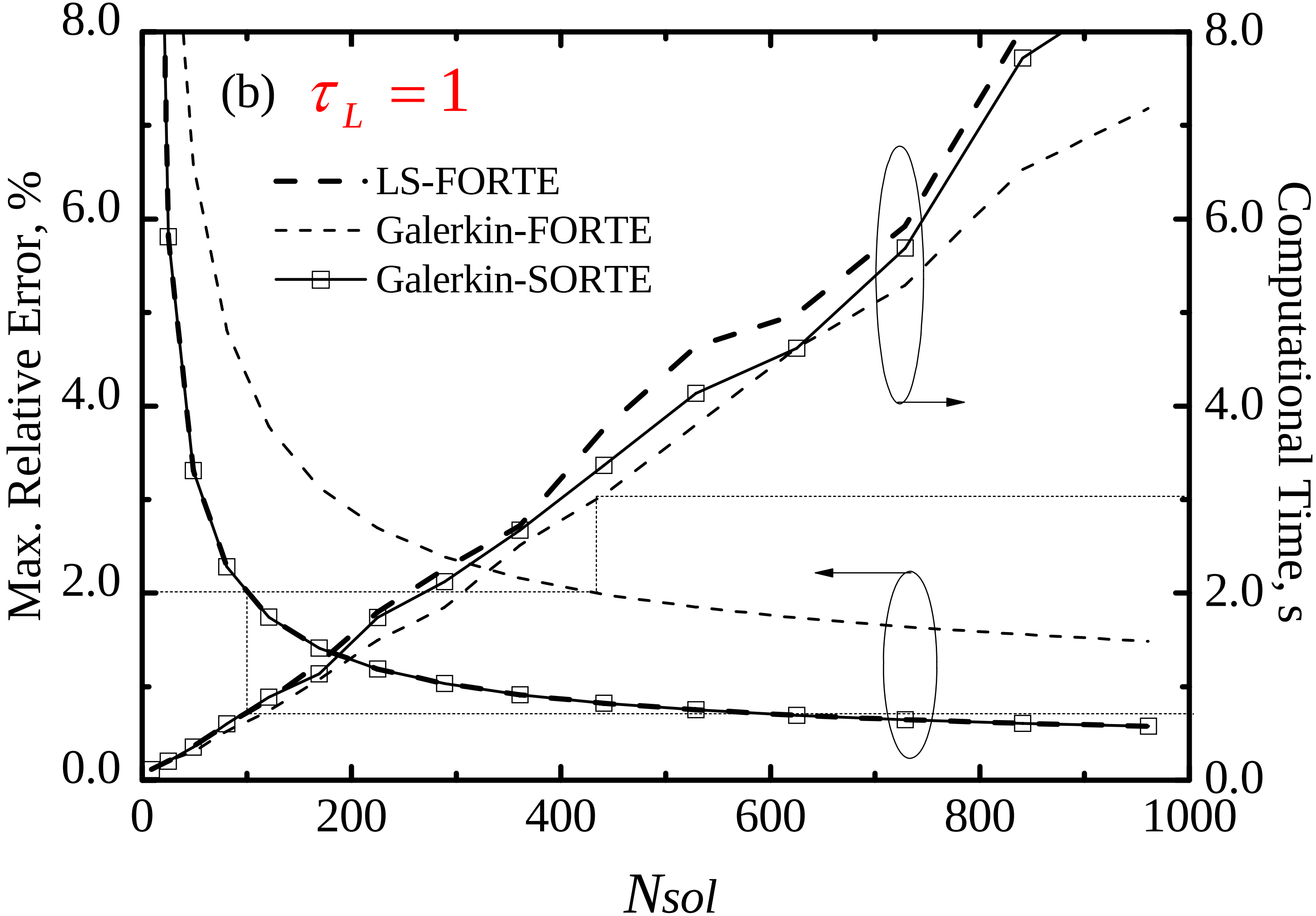
Solid angle: S_8

Solution quantity: bottom wall radiative heat flux

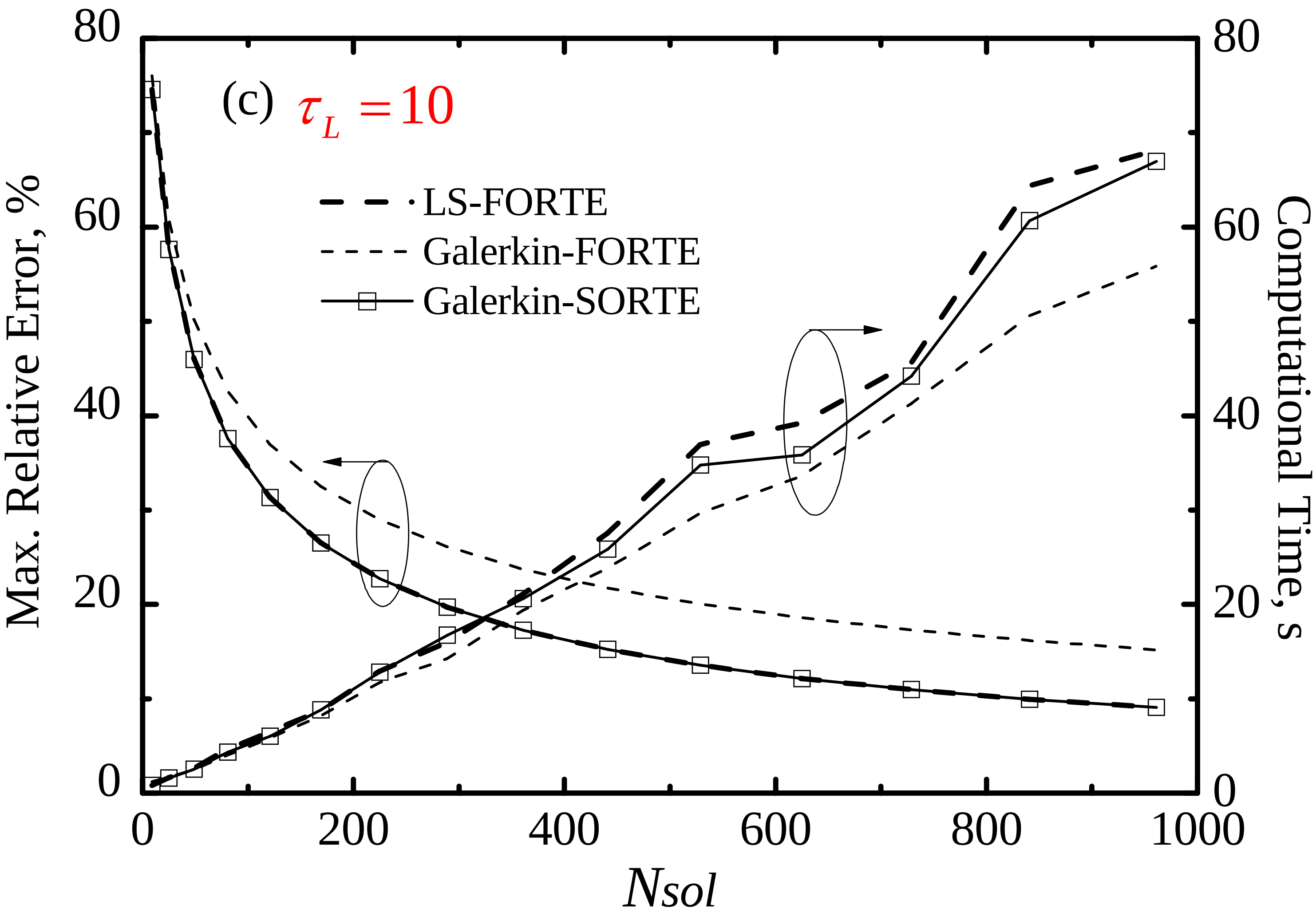
Results and Discussion



Results and Discussion



Results and Discussion



Conclusions

- The accuracy of the FEM based on the SORTE is generally better than that based on the FORTE
- *FEM based on SORTE* is the *most efficient* than the FEMs based on the FORTE.

Questions & comments?

Thanks for your attention!