推导过程很随意. 思路很混乱.

大家在不理解中加深理解吧!

就和着看看吧.

一、排列组合　加法原理、乘法原理。

二、概率之定义

切之定义：前苏联数学家柯尔莫哥洛夫 1933年 公理化定义。

设 Ω 是随机试验 E 之样本空间，对每个事件 A 定义一个实数 $P(A)$ 与之对应。

若函数 $P(A)$ 满足：

(1)、对任何事件 A，均有 $0 \le P(A) \le 1$。（非负性）

(2)、规范性 $P(\Omega) = 1$

(3)、（可列可加性）$A_1, A_2, \cdots, A_n$ 两两互不相容，则有 $A_i \cap A_j = \phi$ $(i \neq j)$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$　　$A_1 \sim A_n$ 互斥完备。

则称 $P(A)$ 为事件 A 的概率。

(三)、名词解释

A. 互斥事件：　$A \cap B = \phi$　两个事件不可能同时发生！

独立事件：$P(A \cap B) = P(A)\,P(B)$　两个事件可能同时发生，但它们之间毫无关系。

(三)、条件概率

$$P(B|A) = \frac{P(A \cap B)}{P(B)} \qquad P(A \cap B) = P(B \cap A)$$

$$\Rightarrow \quad P(A) \cdot \frac{P(A \cap B)}{P(A)} = P(B) \cdot \frac{P(B \cap A)}{P(B)}$$

$$\Rightarrow \quad P(A)\,P(A|B) = P(B)\,P(B|A)$$

若 $P(A \cap B) = P(A)\,P(B)$（独立事件）

$$P(B|A) = \frac{P(B \cap A)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

组合 $\frac{N!}{(N-R)!R!}$

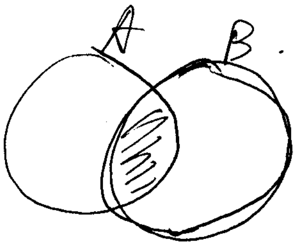
区别 独立事件 / 互斥事件.

条件 $P(B|A)$ 在B条件发生下 出现A的概率.

$$P(B|A)=\frac{P(A\cap B)}{P(B)}$$

互相排斥的 $A\cap B=\phi$.

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$



- 若 $A\cap B=\phi$ 即A与B互斥.

则 $P(A\cup B)=P(A)+P(B)$

$P(A\cap B)=P(A)P(B)$.

$P(A)P(A|B)=P(B)P(B|A)$

$x\in A\cap B \Rightarrow (x\in A)\wedge(x\in B)$

$$P(A)\cdot\frac{P(A\cap B)}{P(A)}=P(B)\frac{P(B\cap A)}{P(B)}$$

会满足乘法原理

$P(A\cap B)=P(A)P(B)$

若

~~(独立事件 $A\cap B=\phi$)~~

$\langle X^n\rangle=\sum_i x_i^n f(x_i)$

- 独立事件不是互相排斥事件.

$\sum_i f(x_i)=1$. X的全部信息.

X S

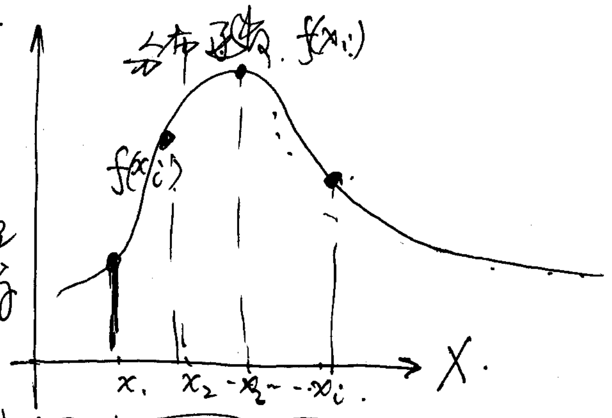
$X: S\mapsto R$. 等价

随机变量 $X: (s_i\in S)\mapsto(x_i\in R)$

—— 映射.

也就是给每个事件一个标记

离散随机变量 $X: (s_i\in S)\mapsto(x_i\in R)$ 或实数



X 连续 很有学问 (例之...)

$\langle X^0 \rangle = \sum_i x_i^0 f(x_i) = \sum_i f(x_i) = 1.$

$\langle X^1 \rangle = \sum_i x_i f(x_i) =$ 平均值.

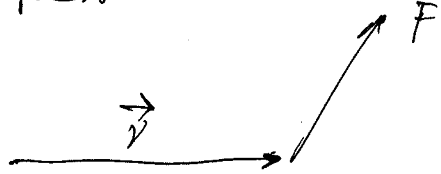
权重　　　是因子.

平均值.

证明:

$$\frac{\sum_i n_i x_i}{\sum n_i} = \frac{\sum_i n_i x_i}{n_{总}} = \sum_i \frac{n_i}{n_{总}} x_i = \sum_i x_i f(x_i)$$

"矩"之概念.



$\vec{M} = \vec{r} \times \vec{F}$ 力矩.

就是在 $\vec{F}$ 上赋予了一个权重 $\vec{r}$.

X 的第 n 阶矩 就是对 x_i^n 赋予一个权重 $f(x_i)$.

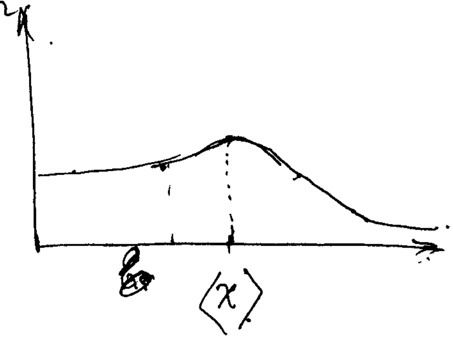
标准偏差. 方差. $(x_i - x_0)^2$

$$\sum_i \frac{(x_i - \langle X \rangle)^2}{n} = \sum_i \frac{x_i^2 - 2x_i \langle X \rangle + \langle X \rangle^2}{n}$$

$$= \frac{\sum_i x_i^2}{n} - 2\langle X \rangle \frac{\sum x_i}{n} + \frac{n\langle X \rangle^2}{n}$$

$$= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2$$

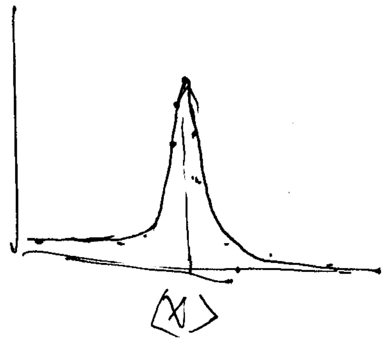
$= \langle X^2 \rangle - \langle X \rangle^2$ 对于平均值的偏离程度.



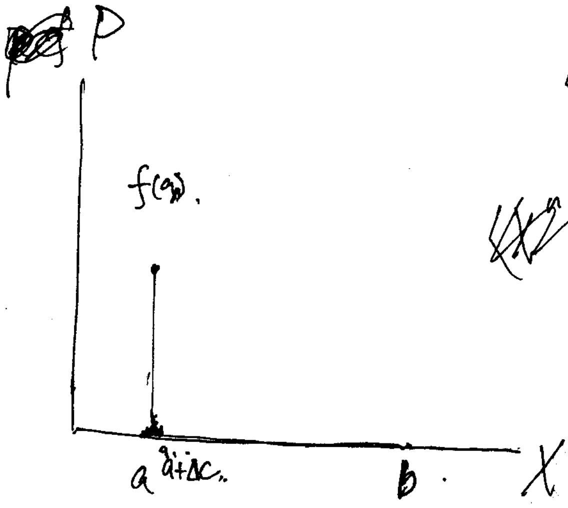
$$= \left(\sum_i x_i^2 f(x_i)\right) - \left(\sum_i x_i f(x_i)\right)^2$$

$$= \sum_i x_i^2 f(x_i) - \sum_i \sum_j x_i x_j f(x_i) f(x_j)$$

$$= \sum_i \left(x_i^2 f(x_i) - \sum_j x_i x_j f(x_i) f(x_j)\right)$$



连续随机变量 连续值的集合.



$$\langle X^n \rangle = \frac{1}{i^n} \frac{d^n \phi_X(k)}{dk^n}\Big|_{k=0}.$$

d

$$f(a_1)\Delta c_1 + f(a_2)\Delta c_2 + \cdots$$

$$\lim_{\substack{i\to\infty \\ \Delta c\to 0}} \sum_i f(a_i)\,\Delta c. \Rightarrow \int_a^b f(x)\,dx = 1.$$

$f(x)$ 是密度函数 ≥ 0.

$$\langle X^n \rangle = \int dx\, x^n f_X(x)$$

特征函数.

$$\phi_X(k) = \langle e^{ikx} \rangle = \int dx\, e^{ikx} f_X(x).$$

$$= \sum_{n=0}^{\infty} \frac{(ik)^n \langle X^n \rangle}{n!}$$

$$e^{ikx} = \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n.$$

$$\phi_X(k) = \int dx\, e^{ikx} f_X(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \int dx\, x^n f_X(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle.$$

$$f_X(x) = \frac{1}{2\pi} \int dk\, e^{-ikx} \phi_X(k)$$

5.16 5.17.

$$\phi_X(k) = \cancel{\langle X^n\rangle}\cdot 1 + \langle X^1\rangle (ik) + \frac{1}{2!}\langle X^2\rangle (ik)^2 + \cdots$$

$$\frac{1}{i}\frac{d\phi_X}{dk}\Big|_{k=0} = 0 + i\langle X^1\rangle + \underbrace{i\frac{1}{2!}\langle X^2\rangle \cdot 2\,(ik) + \cdots\cdots}_{=0}$$

$$\frac{1}{i^2}\frac{d^2\phi_X}{dk^2}\Big|_{k=0} = 0 + 0 + i i\langle X^2\rangle + \underbrace{i i\frac{1}{3!}\langle X^3\rangle \cdot 3\cdot 2\,(ik)}_{k=0}$$

$$\vdots$$

$$\langle X^n\rangle = \frac{1}{i^n}\frac{d^n\phi_X}{dk^n}\Big|_{k=0} \qquad \phi_X(k) = \sum_{n=0}^{\infty}\frac{(ik)^n\langle X^n\rangle}{n!}$$

特征函数

$$\phi_X(k) = e^{\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}C_n(X)}$$ 其中 $C_n(X)$ 为累积量

$$\phi_X(k) = \sum_{n=0}^{\infty}\frac{(ik)^n\langle X^n\rangle}{n!}$$

$$\phi_X(k) = e^{ikC_1(X) + \frac{1}{2!}(ik)^2C_2(X) + \frac{1}{3!}(ik)^3C_3(X) + \cdots}$$

$$\frac{d\phi_X(k)}{dk}\Big|_{k=0} = \left(iC_1(X) + \underbrace{i^2C_2(X)\,k + \frac{i^3}{2!}C_3(X)\,k^2 + \cdots}_{0}\right)e^{\phi}$$

$$\phi_X(k) = e^{iC_1(X)k + \frac{1}{2!}i^2C_2(X)k^2 + \frac{i^3}{3!}C_3(X)k^3 + \cdots}$$

$$\frac{d\phi_X(k)}{dk}\Big|_{k=0} = i\,C_1(X)$$

$c_1(X) = \langle X^1 \rangle$

$$\left.\frac{d^2 \phi_X(k)}{dk^2}\right|_{k=0} = \left(i^2 c_2(X) + \underbrace{i^3 c_3(X) k + \cdots}_{0}\right) e^{\overbrace{\quad\quad}^{0}}$$

$$+ \left(\underbrace{i c_1(X) + i^2 c_2(X) k + \frac{i^3}{2!} c_3(X) k^2 + \cdots}_{0}\right)^2 e^{\overbrace{\quad\quad}^{0}}$$

$$= -c_2(X) - c_1(X) = i^2 \langle X^2 \rangle$$

$$\langle X^2 \rangle = c_2(X) + c_1(X) = c_2(X) + \langle X \rangle$$

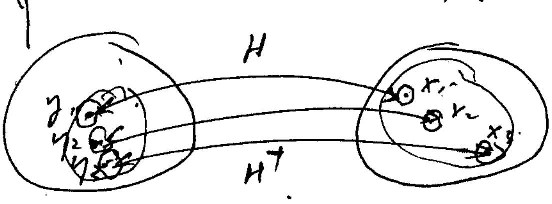
$$\underline{c_2(X) = \langle X^2 \rangle - \langle X \rangle}$$

$f_Y(y) = \int dx\, \delta(y - H(x))\, f_X(x)$

$\cancel{y = H(x)} = f_X(H^{-1}(y))$

X 和 Y

$x = H^{-1}(y)$



一个样品有两个性状 (x_i, y_i) $\longrightarrow$ $\vec{\gamma} = (x_i, x_i, \cdots x_n) \to x$

协方差 $\mathrm{cov}(X, Y) = \iint dx\,dy\,(x - \langle x \rangle)(y - \langle y \rangle) f(x, y)$

(X, Y) 是否可以分离变量 （~~独~~立、耦合）

$\mathrm{cov}(X, Y)$ 协方差.

$= \iint dx\,dy\,(x - \langle X\rangle)(y - \langle Y\rangle) f(x, y)$

$= \iint dx\,dy\,\left[xy - x\langle Y\rangle - y\langle X\rangle + \langle X\rangle\langle Y\rangle\right] f(x, y)$

$= \iint dx\,dy\; xy f(x, y) - \langle Y\rangle \int x \underbrace{\int f(x, y)\, dy}\, dx$

$- \langle X\rangle \int y \int f(x, y)\, dx\, dy + \langle X\rangle\langle Y\rangle \iint dx\,dy\, f(x, y)$

$= \langle X Y\rangle - \langle Y\rangle \int x f_X(x, y)\, dx - \langle X\rangle \int y f_Y(x, y)\, dy$

$+ \langle X\rangle\langle Y\rangle$

$= \langle XY\rangle - \langle Y\rangle\langle X\rangle - \langle X\rangle\langle Y\rangle + \langle X\rangle\langle Y\rangle$

$= \langle XY\rangle - \langle X\rangle\langle Y\rangle$

$\mathrm{cov}(aX+b, cY+d)$

$= \langle (aX+b)(cY+d)\rangle - \langle aX+b\rangle\langle cY+d\rangle$

$= \langle acXY + adX + bcY + bd\rangle - (a\langle X\rangle + b)(c\langle Y\rangle + d)$

$= ac\langle XY\rangle + ad\langle X\rangle + bc\langle Y\rangle + bd - ac\langle X\rangle\langle Y\rangle$

$- ad\langle X\rangle - bc\langle Y\rangle - bd$

$= ac(\langle XY\rangle - \langle X\rangle\langle Y\rangle)$

$\Rightarrow \mathrm{cor}(aX+b, cY+d) = \dfrac{ac(\langle XY\rangle - \langle X\rangle\langle Y\rangle)}{ac\,\sigma_X\,\sigma_Y}$

$\sigma_{aX+b} = a\sigma_X \qquad \sigma_{cY+d} = c\sigma_Y$

$= \mathrm{cor}(X, Y)$

定理（柯西—施瓦茨不等式）.

设随机变量 ξ 和 η 若 $\langle\xi^2\rangle<\infty$. $\langle\eta^2\rangle<\infty$. 则有

$$|\langle\xi\eta\rangle|^2 \le \langle\xi^2\rangle\langle\eta^2\rangle. \ \star$$

证明: 设 t 是一个实变量 $\langle(\xi+t\eta)^2\rangle \equiv g(t)$

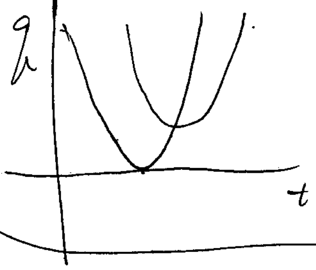
$$g(t)=\langle(\xi+t\eta)^2\rangle=\langle\xi^2+2t\xi\eta+t^2\eta^2\rangle$$

$$=\langle\xi^2\rangle+2t\langle\xi\eta\rangle+t^2\langle\eta^2\rangle.$$

若 $\langle\eta^2\rangle=0$. 则由此 $\langle\xi\eta\rangle=0$, $\star$ 式成立.

若 $\langle\eta^2\rangle>0$, $g(t)\ge 0$ 且 $g(t)$ 是二次多项式, t^2 的系数为 $\langle\xi^2\rangle>0$.

所以 $g=g(t)$ 开口向上. 于是



$$\Delta=4|\langle\xi\eta\rangle|^2-4\langle\xi^2\rangle\langle\eta^2\rangle\le 0.$$

$$|\langle\xi\eta\rangle|^2\le\langle\xi^2\rangle\langle\eta^2\rangle.$$

推论:

$|cor(X,Y)|\le 1$ 且 $|cor(X,Y)|=1$ 充要条件为 存在 a.b 常数

使 $P\{Y=aX+b\}=1$.

C−S不等式

证明: 令 $\eta=X-\langle X\rangle$. $\xi=Y-\langle Y\rangle$.

则 $$[cor(X,Y)]^2=\frac{\{\langle(X-\langle X\rangle)(Y-\langle Y\rangle)\rangle\}^2}{\sigma_X\ \sigma_Y}=\frac{\langle\xi\eta\rangle^2}{\langle\eta^2\rangle\langle\xi^2\rangle}\le 1$$

$$|cor(X,Y)|=1 \Leftrightarrow \langle\xi\cdot\eta\rangle^2=\langle\eta^2\rangle\langle\xi^2\rangle.$$

取 $g(t)=0$. 即 $\langle(\xi+t_0\eta)^2\rangle=g(t_0)=0$.

$$\langle\xi+t_0\eta\rangle=\langle Y-\langle Y\rangle\rangle+t_0\langle X-\langle X\rangle\rangle=0.$$

$\Rightarrow P\{\xi+t_0\eta=0\}=1$

$\Leftrightarrow P\{Y = -t_0 X + (\langle Y \rangle + t_0 \langle X \rangle)\} = 1$

$-t_0 = a$. $\langle Y \rangle + t_0 \langle X \rangle = b$

于是 $\Leftrightarrow P\{Y = aX + b\} = 1$.

分类：

$cor(X, Y) \in (0, 1]$ 正相关.

$cor(X, Y) \in [-1, 0)$ 负相关.

$cor(X, Y) = 0$ 不相关.

不相关不意味着独立.

实验N次. 实现 n_1 次正与$(N-n_1)$次反的概率.

$$P_N(n_1) \equiv \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2}$$ 二项分布.

$$S = \{正(+1),\ 反(-1)\} \qquad \cancel{X} = \{n_1,\ n_2\}$$

二项式分布. $$\sum_{n_1=0}^{N} P_N(n_1) = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} = (p+q)^N = 1.$$

$N!/(n_1!\, n_2!)$. 二项式分布.

$$\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 P_N(n_1) = \sum_{n_1=0}^{N} \frac{N!\, n_1}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}$$

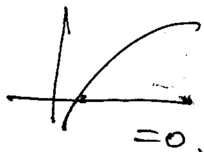
$$= p\frac{\partial}{\partial p}\left[\sum_{n_1=0}^{N} P_N(n_1)\right] \qquad \cancel{\frac{\partial}{\partial p}(p+q)^N = N\cdot\frac{\partial}{\partial p}(p+q)^{N-1}}$$

$$p\frac{\partial}{\partial p}\sum_{n_1=0}^{N} C_N^{n_1} p^{n_1}(1-p)^{n_2}$$

$p+q=$

$$= p\sum_{n_1=0}^{N} C_N^{n_1}\frac{\partial}{\partial p}\left[p^{n_1}(1-p)^{n_2}\right]$$

$$= p\sum_{n_1=0}^{N} C_N^{n_1}\left[n_1 p^{n_1-1}(1-p)^{n_2} - n_2 p^{n_1}(1-p)^{n_2-1}\right]$$



$$= p\sum_{n_1=0}^{N} C_N^{n_1} p^{n_1-1}(1-p)^{n_2} - p\sum_{n_1=0}^{N} n_2 p^{n_1}(1-p)^{n_2-1}.$$

$=0.$

$\ln 1$

$$= \sum_{n_1=0}^{N} C_N^{n_1} p^{n_1}(1-p)^{n_2} - \sum_{n_1=0}^{N} n_2 p^{n_1+1}(1-p)^{n_2-1}.$$

$$\frac{\partial}{\partial p} e^{N\ln(p+q)} = e^{N\ln(p+q)}\cdot N\cdot\frac{1}{p+q}\cdot 1.$$

$$= N e^{N\ln(p+q)}$$

斯特林公式.

N很大,并且 pN 也很大. 也就是说 p不是太小.

$$n_1! \simeq \sqrt{2\pi n_1}\left(\frac{n_1}{e}\right)^{n_1}.$$

$$n_1! \approx \sqrt{2\pi n_1}\left(\frac{n_1}{e}\right)^{n_1}.$$

$\langle n_1^2 \rangle$

$$\sum_{n_1=0}^{N} \frac{N!\, n_1}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}.$$

$$\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 P_N(n_1).$$

① ② ③

$$p\frac{\partial}{\partial p}\sum_{n_1=0}^{N} P_N(n_1) = p\frac{\partial}{\partial p}\sum_{n_1=0}^{N} \frac{N!}{n_1!\,n_2!} p^{n_1} q^{n_2}.$$

$$= p\sum_{n_1=0}^{N} \frac{N!\, n_1}{n_1!\,n_2!} p^{n_1-1} q^{n_2} = \sum_{n_1=0}^{N} \frac{N!\, n_1}{n_1!\,n_2!} p^{n_1} q^{n_2}.$$

$$\frac{\partial}{\partial p}(p+q)^N = \frac{\partial}{\partial p} e^{N\ln(p+q)}$$

$$= e^{N\ln(p+q)} \cdot N \cdot \frac{1}{p+q} \cdot 1 = N\frac{e^{N\ln(p+q)}}{p+q} = N\frac{e^0}{1} = N.$$

$(p+q=1)$

$$n_1! \approx \sqrt{2\pi n_1}\cdot\left(\frac{n_1}{e}\right)^{n_1}.$$

$n_1! \sim$

$$= \sqrt{2\pi n_1}\; e^{n_1(\ln n_1 - \ln e)}.$$

$$= \sqrt{2\pi n_1}\; e^{n_1(\ln n_1 - 1)}.$$

$$= \sqrt{2\pi n_1}\left(e^{n_1 \ln n_1} - e^{n_1}\right).$$

$$\langle n_1^2\rangle = \sum_{n_1=0}^{N} n_1^2 P_N(n_1) = \sum_{n_1=0}^{N} \frac{N!\, n_1^2}{n_1!\, n_2!} p^{n_1} q^{n_2}$$

=

$$p^2 \frac{\partial^2}{\partial p^2} \sum_{n_1=0}^{N} P_N(n_1)$$

$$= p^2 \frac{\partial^2}{\partial p^2} \sum_{n_1=0}^{N} \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2}$$

$$= p^2 \sum_{n_1=0}^{N} \frac{N!}{n_1!\, n_2!} q^{n_2} \frac{\partial^2 p^{n_1}}{\partial p^2}$$

$$= p^2 \sum_{n_1=0}^{N} \frac{N!}{n_1!\, n_2!} q^{n_2}\, n_1(n_1-1)\, p^{n_1-2}$$

$$= \sum_{n_1=0}^{N} \frac{N!\, n_1^2}{n_1!\, n_2!} p^{n_1} q^{n_2} - \sum_{n_1=0}^{N} \frac{N!\, n_1}{n_1!\, n_2!} p^{n_1} q^{n_2}$$

$$= \sum_{n_1=0}^{N} \frac{N!\, n_1^2}{n_1!\, n_2!} p^{n_1} q^{n_2} - p \frac{\partial}{\partial p} \sum_{n=0}^{N} P_N(n_1)$$

$$= \langle n_1^2\rangle - pN$$

$$p^2 \frac{\partial^2}{\partial p^2} e^{N\ln(p+q)}$$

$$Np^2 \frac{\partial}{\partial p}\left(e^{N\ln(p+q)} \frac{1}{p+q}\right)$$

$$Np^2 \frac{Ne^{N\ln(p+q)} - e^{N\ln(p+q)}}{(p+q)^2}$$

$$Np^2 \frac{(N-1)e^{N\ln(p+q)}}{(p+q)^2}$$

$$N^2p^2 - Np^2$$

$$\langle n_1^2\rangle - pN$$

$$\langle n_1^2\rangle - pN = N^2p^2 - Np^2$$

$$\langle n_1^2\rangle = (Np)^2 + Np(1-p) = (Np)^2 + Npq$$

$$\sigma_N^2 = \langle n_1^2\rangle - \langle n_1\rangle^2 = (Np)^2 + Npq - (pN)^2$$

$$= Npq$$

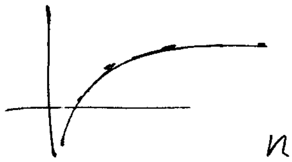
$$\sigma_N = \sqrt{\sigma_N^2} = \sqrt{Npq}$$

$$\frac{\sigma_N}{\langle n_1\rangle} = \frac{\sqrt{Npq}}{Np}$$

$$= \frac{1}{\sqrt{N}} \sqrt{\frac{q}{p}}$$

$\sqrt{2\pi n_1}\, e^{n_1 \ln n_1} - \sqrt{2\pi n_1}\, e^{n_1}$. $\lim_{n_1\to\infty}$.

$\frac{d \ln n_1}{d n_1} = \frac{1}{n_1}$.



$n_1! \approx \sqrt{2\pi n_1} \left(\frac{n_1}{e}\right)^{n_1}$.

$P_N(n_1) = \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2}$.

$= \frac{N!}{n_1!\,(N-n_1)!} p^{n_1} q^{(N-n_1)}$.

$= \frac{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N p^{n_1} (1-p)^{N-n_1}}{\sqrt{2\pi n_1}\sqrt{2\pi (N-n_1)}\left(\frac{n_1}{e}\right)^{n_1}\left(\frac{N-n_1}{e}\right)^{N-n_1}}$.

故在此区域中

$P_N(n_1) = \frac{1}{\sqrt{2\pi N}}$

$\begin{cases} [N-(n_1 \pm \frac{1}{2})] \\ (n_1+\frac{1}{2}) \end{cases}$ $\begin{pmatrix} N-(Np-\frac{1}{2}) \\ Np+\frac{1}{2} \end{pmatrix}$

Np 很大. 所以 $Np \pm \frac{1}{2} \to Np$.

$\left(\frac{dP_N}{dn_1}\right)_{n_1=\langle n_1\rangle} = 0$.

$$P_N(n_1) = \frac{1}{\sqrt{2\pi N}} \exp\left[-(n_1)\ln\left(\frac{n_1}{N}\right) - (N-n_1)\ln\left(\frac{N-n_1}{N}\right) + n_1\ln p + (N-n_1)\ln(1-p)\right]$$

考查一下它的极大值处是否在 $n_1 \stackrel{?}{=} \langle n_1\rangle = Np$ 平均值

$\frac{dP_N}{dn_1} = 0 \Rightarrow$ 得出 n_1 即可

$$\frac{1}{\sqrt{2\pi N}}\exp\{\}\left\{(-1)\ln\frac{n_1}{N} - n_1\frac{N}{n_1}\frac{1}{N} - (-1)\ln\left(\frac{N-n_1}{N}\right) - (N-n_1)(-1)\frac{N}{(N-n_1)}\frac{1}{N} + \ln p + (-1)\ln(1-p)\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\times\left\{-\ln\frac{n_1}{N} - 1 + \ln\frac{N-n_1}{N} + 1 + \ln p - \ln(1-p)\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\times\left\{-\ln\frac{n_1}{N} + \ln\frac{N-n_1}{N} + \ln p - \ln(1-p)\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\times \ln\frac{N(N-n_1)p}{n_1 N(1-p)} = 0$$

$\cancel{\Rightarrow \frac{n_1(N-n_1)p}{N^2(1-p)} = 1}$ $\quad$ $\cancel{(n_1N - n_1^2)p = N^2 - Np}$

$\cancel{n_1 n_2 p = N^2 q}$

$\cancel{n_1 N q p = N^2 q}$

$$\cancel{\frac{N p (N-n_1)p^2}{N^2(1-p)}}$$

$$\frac{(N-n_1)p}{n_1(1-p)} = 1$$

$$(N-n_1)p = n_1(1-p)$$

$$Np - n_1 p = n_1 - n_1 p \Rightarrow n_1 = Np = \langle n_1\rangle$$

$$P_N(n_1) = \frac{1}{\sqrt{2\pi N}} \exp\left\{-n_1\ln\left(\frac{n_1}{N}\right) - (N-n_1)\ln\left(\frac{N-n_1}{N}\right) + n_1\ln p + (N-n_1)\ln(1-p)\right\}$$

一阶

$$\frac{dP_N}{dn_1} = \frac{1}{\sqrt{2\pi N}}\exp\{\}\ \ln\frac{(N-n_1)p}{n_1(1-p)} = \frac{1}{\sqrt{2\pi N}}\exp\{\}\cdot\left\{\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right\} = 0.$$

$$n_1 = Np = \langle n_1\rangle.$$

二阶

$$\frac{d^2P_N}{dn^2} = \frac{1}{\sqrt{2\pi N}}\left\{\exp\{\}\left\{\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right\}^2 + \exp\{\}\left\{(-1)\frac{1}{N-n_1} - \frac{1}{n_1}\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\left\{\left(\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right)^2 - \frac{1}{N-n_1} - \frac{1}{n_1}\right\} = \frac{1}{\sqrt{2\pi N}}\exp\{\}\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right)$$

三阶

$$\frac{d^3P_N}{dn^3} = \frac{1}{\sqrt{2\pi N}}\left\{\exp\{\}\left\{\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right\}\left\{\left(\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right)^2 - \frac{1}{N-n_1} - \frac{1}{n_1}\right\}\right.$$

$$\left. + \exp\{\}\left\{2\left(\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right)\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right) - (-1)\frac{1}{(N-n_1)^2}(-1) - (-1)\frac{1}{n_1^2}\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\left\{\left[\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right]\left\{\left[\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right]^2 - \frac{1}{N-n_1} - \frac{1}{n_1}\right\}\right.$$

$$\left. + 2\left[\ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p}\right]\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right) - \frac{1}{(N-n_1)^2} + \frac{1}{n_1^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi N}}\exp\{\}\left[-\frac{1}{(N-n_1)^2} + \frac{1}{n_1^2}\right].$$

在 $n_1 = \langle n_1\rangle$ 处泰勒展开 设 $n_1 - \langle n_1\rangle = \varepsilon$.

$$P_N(n_1) = \frac{1}{0!}P_N(\langle n_1\rangle) + \frac{1}{1!}\cdot 0\cdot\varepsilon + \frac{1}{2!}P_N(\langle n_1\rangle)\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right)\varepsilon^2 + \frac{1}{3!}P_N(\langle n_1\rangle)\left[\frac{1}{n_1^2} - \frac{1}{(N-n_1)^2}\right]\varepsilon^3 + \cdots$$

$$= P_N(\langle n_1\rangle)\cdot\left\{1 + 0\varepsilon + \frac{1}{2!}\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right)\varepsilon^2 + \frac{1}{3!}\left[\frac{1}{n_1^2} - \frac{1}{(N-n_1)^2}\right]\varepsilon^3 + \cdots\right\}$$

$$= P_N(\langle n_1\rangle)\cdot\left\{1 + \frac{1}{2}\left(-\frac{1}{N-n_1} - \frac{1}{n_1}\right)\varepsilon^2 + \frac{1}{6}\left[\frac{1}{n_1^2} - \frac{1}{(N-n_1)^2}\right]\varepsilon^3 + \cdots\right\}$$

~~$e^{\ln 1+\frac{1}{2}}$~~ 0阶 $B_0 = 0.$

~~$\exp\{\ln 1+\frac{1}{2}$~~

对指数字量在 $n_1=\langle n_1\rangle$ 处展开

$$P_N(n_1) = \frac{1}{\sqrt{2\pi N}} \exp\left\{ \underbrace{-n_1 \ln\left(\frac{n_1}{N}\right) - (N-n_1)\ln\left(\frac{N-n_1}{N}\right) + n_1\ln p + (N-n_1)\ln(1-p)} \right\}$$

一阶

$$B_1 = \frac{d\{\}}{dn_1}\Big|_{n_1=\langle n_1\rangle} = \ln(N-n_1) - \ln n_1 + \ln\frac{p}{1-p} = 0.$$

二阶

$$B_2 = \frac{d^2\{\}}{dn_1^2}\Big|_{n_1=\langle n_1\rangle} = -\frac{1}{N-n_1} - \frac{1}{n_1} = -\frac{1}{N-Np} - \frac{1}{Np} = -\frac{1}{N}\frac{p+q}{pq} = -\frac{1}{Npq}$$

三阶

$$B_3 = \frac{d^3\{\}}{dn_1^3}\Big|_{n_1=\langle n_1\rangle} = \frac{1}{n_1^2} - \frac{1}{(N-n_1)^2} = \frac{1}{N^2p^2} - \frac{1}{(N-Np)^2} = \frac{1}{N^2}\left(\frac{1}{p^2}-\frac{1}{q^2}\right).$$

$$= \frac{1}{N^2p^2q^2}(q^2-p^2). \qquad (q^2-p^2)^2<1.$$

$$B_3^2 = \frac{1}{N^4p^4q^4}(q^2-p^2)^2 \Rightarrow \boxed{|B_3| < \frac{1}{N^2p^2q^2}}$$

对于高阶 $|B_k| < \frac{1}{(Npq)^{k-1}}.$

于是 $P_N(n_1) = \frac{1}{\sqrt{2\pi N}}\exp\left\{\frac{1}{0!}B_0\varepsilon^0 + \frac{1}{1!}B_1\varepsilon^1 + \frac{1}{2!}B_2\varepsilon^2+\frac{1}{3!}B_3\varepsilon^3+\cdots\right\}$

$$= \frac{1}{\sqrt{2\pi N}}\exp\left\{0+0+\frac{1}{2} B_2\varepsilon^2+\frac{1}{6} B_3\varepsilon^3+\cdots\right\}.$$

其中 $B_2 = -\frac{1}{Npq}$ $\quad B_3 = \frac{1}{N^2p^2q^2}(q^2-p^2)$ $\quad |B_3| < \frac{1}{N^2p^2q^2}$ $\cdots$

且 $|B_k| < \frac{1}{(Npq)^{k-1}}$

分析. 如果 $\varepsilon^2 \ll Npq$. 则 ε^2 以高次可以忽略.

于是 指去高阶项.（3次及更以上，保留到2次）

$$P_N(n_1) = \frac{1}{\sqrt{2\pi N}}\, e^{-\frac{1}{2}\frac{1}{Npq}(n_1 - Np)^2}$$

现在的问题是 $\boxed{\sum_i^N P_N(n) \overset{?}{=} 1}$

我们必须尊守概率的归一化.

现在已经是连续的了.

$$I = \int_{-\infty}^{+\infty} P_N(n_1)\, dn_1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi N}} e^{-\frac{(n_1-Np)^2}{2Npq}}\, dn_1$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi N}}\, e^{-\frac{(n_1-Np)^2}{2Npq}}\, d(n_1 - Np)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi N}}\, e^{-\frac{N_1^2}{2Npq}}\, dN_1$$

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi N}}\right)^2 e^{-\frac{x^2+y^2}{2Npq}}\, dx\, dy$$

$$= \iint \frac{1}{2\pi N}\, e^{-r^2/2Npq}\, r\, dr\, d\theta = 1$$

$$\cancel{= \iint \frac{1}{2\pi N} e^{r^2}}$$

$$= \frac{1}{N}\int_0^\infty r e^{-\frac{r^2}{2Npq}}\, dr = \frac{1}{N}\int_0^\infty -2 e^{-\frac{r^2}{2Npq}}\, d\left(\frac{-r^2}{2Npq}\right)$$

$$= \frac{-2}{N}\int_0^\infty \qquad 2Npq$$

$$\Rightarrow \quad P_N(n_1) = \frac{1}{\sqrt{2\pi}\sqrt{Npq}} \exp\left\{-\frac{1}{2}\frac{(n_1-\langle n_1\rangle)^2}{\sigma_N^2}\right\}.$$

$$P_N(n_1) = \frac{1}{\sqrt{2\pi}\,\sigma_N} \exp\left\{-\frac{1}{2}\frac{(n_1-\langle n_1\rangle)^2}{\sigma_N^2}\right\}$$

$$P_N(n_1) = \frac{1}{\sqrt{2\pi N}} \exp\Big\{ -n_1 \ln\left(\frac{n_1}{N}\right) - (N-n_1) \ln \frac{N-n_1}{N}$$
$$+ n_1 \ln p + (N-n_1)\ln(1-p) \Big\}.$$

$$\frac{dP_N}{dn_1} = \frac{1}{\sqrt{2\pi N}} \exp\{\}\cdot\Big\{ -\ln\frac{n_1}{N} - \cancel{n_1 \frac{N}{n_1}\cdot\frac{1}{N}} - (-1)\ln\frac{N-n_1}{N} - \cancel{(N-n_1)\frac{N}{N-n_1}\cdot\frac{1}{N}\cdot(-1)}$$
$$+ \ln p + (-1)\ln(1-p) \Big\}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\{\}\cdot\Big\{ \ln\frac{N-n_1}{N} - \ln\frac{n_1}{N} + \ln p - \ln(1-p) \Big\}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\{\}\cdot\Big\{ \ln\frac{N-n_1}{n_1} + \ln p - \ln(1-p) \Big\}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\{\}\cdot\Big\{ \ln\frac{N-n_1}{n_1} + \ln\frac{p}{1-p} \Big\}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\{\}\cdot \ln\frac{(N-n_1)\,p}{n_1(1-p)} = 0.$$

$$(N-n_1)\,p = n_1(1-p) \Rightarrow n_1 = pN = \langle n_1\rangle$$

$$\cancel{(N-n_1)(1-p) = n_1 p = N - Np - n_1 + n_1 p = n_1 p}$$

$$\cancel{n_1 = N(1-p)} \quad \cancel{\varepsilon = n_1 - \langle n_1\rangle}$$

$$P_N(n_1) = \frac{1}{0!} P_N(\langle n_1\rangle)\,\varepsilon^0 + \frac{1}{1!} \left.\frac{dP_N}{dn}\right|_{\langle n_1\rangle} \varepsilon + \cdots$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$\lim_{p\to 0}(1-p)^{a/p} = \lim_{p\to 0}\left(1+\frac{-1}{(\frac{1}{p})}\right)^{a\cdot(\frac{1}{p})} = \left\{\lim_{p\to 0}\left[\left(1+\frac{-1}{(\frac{1}{p})}\right)^{\frac{1}{p}}\right]\right\}^a$$

$$= (e^{-1})^a = e^{-a}.$$

泊松分布

泊松分布归一化到1.

$$\sum_{n_1=0}^{\infty} a^{n_1}e^{-\alpha}$$

$$P_N(n_1) \equiv \frac{N!}{n_1!\,n_2!}p^{n_1}q^{n_2}$$

$$= \frac{N!}{n_1!\,(N-n_1)!}p^{n_1}q^{n_2}$$

$$= \frac{N^{n_1}}{n_1!}p^{n_1}e^{-a} = \frac{a^{n_1}e^{-a}}{n_1!}$$

$$\sum_{n_1=0}^{\infty}\frac{a^{n_1}e^{-a}}{n_1!} = e^{-a}\sum_{n_1=0}^{\infty}\frac{a^{n_1}}{n_1!} = e^{-a}e^{a} = 1$$ ✓

归一化

泊松分布

$\frac{N!}{(N-n_1)!}$ 利用斯特林公式 $\sim N^{n_1}$ ✓

若每一步的长度为 l.

$x = ml$ 考虑，比 l 大得多的区间 Δx

上下只差 N^{n_1}

$$\frac{N!}{(N-n_1)!} \approx \frac{\sqrt{2\pi N}}{\sqrt{2\pi (N-n_1)}}\frac{\left(\frac{N}{e}\right)^{N}}{\left(\frac{N-n_1}{e}\right)^{N-n_1}} = \sqrt{\frac{N}{N-n_1}}\,\frac{e^{N\ln N}}{e^{(N-n_1)\ln(N-n_1)}\,e^{n_1}} \sim N^{n_1}$$

$N\to\infty$

$$\lim_{N\to\infty}\sqrt{\frac{N}{N-n_1}}\,e^{N\ln N-(N-n_1)\ln(N-n_1)-n_1}$$

$$=\lim_{N\to\infty}\frac{e^{N\ln N}\cdot(\ln N+1)}{e^{n_1}\,e^{(N-n_1)\ln(N-n_1)}\cdot(\ln(N-n_1)-1)}$$

J.6

随机行走.

① 什么是随机行走.

② 满足什么分布 $\longrightarrow$ 二项式分布

用什么近似 $\begin{cases} \text{高斯分布} ? & N\to\infty \quad Np \text{ 很大}. \\ \text{泊松分布} ? & N\to\infty . \quad p\to 0. \end{cases}$

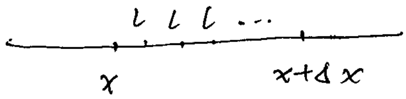
选择高斯分布.

右 n_1 左 n_2. 总 $N=n_1+n_2$. 净位移 $m=n_1-n_2$ 真正的位置

于是 $m=2n_1-N \Rightarrow n_1=\frac{m+N}{2}$ $m\in[-N,N]$

$N\to\infty$ 走过很多步之后. n_1 代入高斯分布函数

$$P_N(m)=\left(\frac{2}{\pi N}\right)^{1/2}\exp\left(-\frac{m^2}{2N}\right).$$ 服从高斯分布



$x=ml.$

概率 = 概率密度 × 宽度.

$$P_N(x)\,\Delta x = P_N(m)\,(\Delta x/2l).$$

单位时间走 n 步.
$N=nt$
每一步长为 l.
$x=Nl=nlt.$

$$P_N(x)=\frac{1}{(2\pi N l^2)^{1/2}}\exp\left(\frac{-x^2}{2Nl^2}\right)$$

$$P(x,t)\,\Delta x=\frac{1}{2(\pi D t)^{1/2}}\exp\left(\frac{-x^2}{4Dt}\right)\Delta x$$

$$D=\frac{1}{2}nl^2$$