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BOCHNER IDENTITY FOR HARMONIC MAPS

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ABSTRACT. Considered in this paper is one of the most important formulas for a harmonic map.

Theorem 1. *If $u \in C^2(M;N)$ is a harmonic map, then in a local coordinate system, there holds*

$$\Delta_g e(u) = |\nabla du|^2 + R^M_{\alpha\beta} u_\alpha u_\beta - R^N_{ijkl}(u) u^i_\alpha u^j_\beta u^k_\alpha u^l_\beta.$$

Proof. Fix an $x_0 \in M$, let (x_α) be a normal coordinate system around x_0, then

$$\begin{aligned}
\Delta_g e(u) &= \partial_\beta \left(u_\alpha, u_{\beta\alpha}\right) \\
&= \left|u_{\alpha\beta}\right|^2 + \left\langle u_\alpha, u_{\beta\alpha,\beta}\right\rangle \\
&= \left|u_{\alpha\beta}\right|^2 + \left\langle u_\alpha, R^M_{\alpha\beta} u_\beta + u_{\beta\beta,\alpha}\right\rangle \\
&= \left|u_{\alpha\beta}\right|^2 + R_{\alpha\beta} u_\alpha u_\beta + \left\langle u_\alpha \left(\Delta_g u\right)_\alpha \right\rangle;
\end{aligned}$$

$$\begin{aligned}
\left|u_{\alpha\beta}\right|^2 &= \left|P(u)(u_{\alpha\beta})\right|^2 + \left|A(u)\left(u_\alpha, u_\beta\right)\right|^2 \\
&= |\nabla du|^2 + \left|A(u)(u_\alpha, u_\beta)\right|^2;
\end{aligned}$$

$$\begin{aligned}
\left\langle u_\alpha, \left(\Delta_g u\right)_\alpha \right\rangle &= -\left\langle u_\alpha, \left(A(u)\left(\nabla u, \nabla u\right)\right)_\alpha \right\rangle \\
&= \left\langle \Delta_g u, A(u)\left(\nabla u, \nabla u\right)\right\rangle \\
&= -\left\langle A(u)(\nabla u, \nabla u), A(u)(\nabla u, \nabla u)\right\rangle
\end{aligned}$$

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$$= \; -\left\langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta)\right\rangle .$$

Thus

$$\Delta_g e(u) = |\nabla du|^2 + R^M_{\alpha\beta} u_\alpha u_\beta - R^N_{ijkl}(u) u^i_\alpha u^j_\beta u^k_\alpha u^l_\beta,$$

by Gauss-Kodazzi equations. □

Proposition 2. *Let*

1. (M, g) *be a closed manifold with* $Ric^M \geq 0$*;*
2. *the sectional curvature of* N*,* $K^N \leq 0$*.*

Then

1. *any harmonic map* $u \in C^2(M; N)$ *is totally geodesic.*
2. *If* $Ric^M > 0$ *at some point in* M*, then* u *is constant.*
3. *If* $K^N < 0$*, then either* u *is constant or* $u(M)$ *lies in a closed geodesic.*

Proof. 1.

$$\begin{aligned} & \Delta_g e(u) \geq 0 \\ \Rightarrow \quad & e(u) \text{ is subharmonic in } M \\ \Rightarrow \quad & e(u) \text{ is constant } \big(\text{maximum principle}\big). \end{aligned}$$

2.

$$\begin{aligned} & Ric^M(x_0) > 0 \\ \Rightarrow \quad & \nabla u(x_0) = 0 \\ \Rightarrow \quad & e(u) \equiv 0 \\ \Rightarrow \quad & u \text{ is constant.} \end{aligned}$$

3.

$$K^N < 0$$

$$\Rightarrow \quad \text{the linear span of } \left\{u^1, \cdots, u^n\right\} \text{ is at most of one dimension}$$

$$\Rightarrow \quad u(M) \begin{cases} \text{is a point} \\ \text{or lies in a closed geodesic} \end{cases} \text{in } N.$$

□

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