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AN EXISTENCE THEOREM FOR STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS UNDER MODIFIED DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. Four types of boundary conditions are considered for the stationary compressible Navier-Stokes equations. This is [1, Page 121], and is delivered on Dec. 11th, 2010.

1. **Introduction.** In this short paper, we consider the following stationary incompressible Navier-Stokes equations:

$$\left.\begin{aligned} &\mathrm{div}\ (\rho u) = 0 \\ &\mathrm{div}\ \ (\rho u \otimes u) - \mu \Delta u - \xi \nabla \mathrm{div}\ u + \nabla\, (a\rho^\gamma) = \rho f + g \end{aligned}\right\} \text{in } \Omega, \tag{1}$$

under boundary condition

(BC1) $u \cdot n = 0$ on $\partial\Omega$; or

(BC2) $\left.\begin{aligned} &\mathrm{curl}\ u = 0\ (N = 2) \\ &\mathrm{curl}\ u \times n = 0\ (N = 3) \end{aligned}\right\}$ on $\partial\Omega$; or

(BC3) $(d \cdot n + Au) \times n = 0$ on $\partial\Omega$, with

$$d = \frac{\nabla u + (\nabla u)^t}{2} \text{ is the deformation tensor,}$$

$$A \text{ is a positive-definite matrix,}$$

$$(Qx + u_0) \cdot n(x) \neq 0 \text{ on } \partial\Omega,\ \forall \text{ antisymmetric } N \times N \text{ matrix } Q \text{ and } u_0 \in \boldsymbol{R}^N,\ \text{unless } Q = 0, u_0 = 0,$$

$$\xi > \frac{N-2}{N}\mu;\ \text{or}$$

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(BC4) $\left(\frac{\partial u}{\partial n}+Au\right)\times n=0$ on $\partial\Omega$, with

A is a nonpositive-definite (not necessarily symmetric) matrix,

$$\xi\geq-\frac{\mu}{N}.$$

2. **Existence Result.** The main result now reads

Theorem 1. *Let $N=2$ or $N=3$, $\gamma>0$, and $p=p(\gamma,N)$ is large enough. Then there exists a continuum $C\left(\subset L^q\times W^{1,q},\ 1\leq q<\infty\right)$ of solutions of (1) under (BC1), or (BC2), or (BC3), or (BC4), satisfying*

1. *$(0,u_0)\in C$, with u_0 solves*

$$\begin{cases}-\mu\Delta u_0-\xi\nabla div\ u_0=0, & in\ \Omega,\\ u_0\ satisfies\ (BC1),\ or\ (BC2),\ or\ (BC3),\ or\ (BC4).\end{cases}$$

2. *$\forall\ M\in[0,\infty)$, $\exists\ (\rho,u)\in C$, such that $\int\rho^p=M$.*

Proof. 1. We approximate (1) by

$$\begin{cases}\text{div }(\rho u)=0,\ \rho\geq0,\ \text{in }\Omega,\ \int_\Omega\rho^p=M,\\ \text{div }(\rho u\otimes u)-\mu\Delta u-\xi\nabla\text{div }u+\nabla(a\rho^\gamma+\alpha\rho^p)=\rho f+g,\ \text{in }\Omega,\\ u\text{ satisfies (BC1), or (BC2), or (BC3), or (BC4),}\end{cases}\tag{2}$$

with $\alpha\in(0,1]$, and $p>3$ is large enough.

2. Notice that the proof of
   (a) the existence of a solution continuum C_α to (2); and
   (b) the passage to limit $C_\alpha\to_\alpha C$;
   are exactly the same as in [2].
3. Thus we need only to show a prior that

$$\left.\begin{array}{l}(\rho,u,M)\in C_\alpha\\ 0\leq M\leq R<\infty\end{array}\right\}\Rightarrow\begin{cases}\rho\text{ bdd in }L^\infty,\ u\text{ bdd in }W^{1,q},\\ \text{div }u-\frac{a}{\mu+\xi}\rho^\gamma-\frac{\alpha}{\mu+\xi}\rho^p\text{ bdd in }W^{1,q},\quad\textbf{uniformly in }\alpha\in(\mathbf{0},\mathbf{1}].\\ \text{curl }u\text{ bdd in }W^{1,q},\ \forall\ 1\leq q<\infty,\end{cases}$$

For this purpose, we shall consider $N = 3$ ($N = 2$ being similar and simple). Our strategy is the usual (by now) **bootstrap argument involving the Hodge decomposition**.

Write $(2)_2$ in the form

$$\nabla\left\{\operatorname{div} u - \frac{a}{\mu+\xi}\rho^\gamma - \frac{\alpha}{\mu+\xi}\rho^p\right\} + \frac{\mu}{\mu+\xi}\operatorname{curl}\operatorname{curl} u = (\rho u\cdot\nabla)\, u + \cdots . \tag{3}$$

We use (3) to bootstrap the regularity of u, and then that of ρ by $(2)_2$. Take first $\rho \in L^{p_i}$, $\nabla u \in L^{q_i}$, with $p_0 = p$, $q_0 = 2$, we have

$$\nabla\left\{\operatorname{div} u - \frac{a}{\mu+\xi}\rho^\gamma - \frac{\alpha}{\mu+\xi}\rho^p\right\},\ D\operatorname{curl} u\ \in L^{r_i},\ \frac{1}{r_i} = \frac{1}{p_i} + \left(\frac{1}{q_i} - \frac{1}{3}\right) + \frac{1}{q_i};$$

$$Du,\ a\rho^\gamma + \alpha\rho^p\ \in L^{q_{i+1}},\ \frac{1}{q_{i+1}} = \frac{1}{r_i} - \frac{1}{3} = \frac{1}{p_i} + \frac{2}{q_i} - \frac{2}{3}\ \big(\text{by } (4)\big).$$

Notice that $p_{i+1} = p_i = p$, since we want to get the uniform bounds (independent of α). Thus

$$\begin{aligned}\frac{1}{q_{i+1}} &= \frac{1}{p_i} + \frac{2}{q_i} - \frac{2}{3} = 2^{i+1}\frac{1}{q_0} + \left(\frac{1}{p} - \frac{2}{3}\right)\left(1 + 2 + \cdots + 2^i\right)\\ &= 2^i + \left(\frac{1}{p} - \frac{2}{3}\right)(2^{i+1} - 1) = 2^i\left[-2\left(\frac{2}{3} - \frac{1}{p}\right) + 1\right] + \frac{2}{3} - \frac{1}{p}\\ &< \frac{1}{3},\ \text{if } i \text{ large.}\end{aligned}$$

Hence $Du \in L^{q_{i+1}>3} \Rightarrow u \in L^\infty$. From then on, we may bootstrap as

$$\frac{1}{q_{i+1}} = \left(\frac{1}{p} + \frac{1}{q_i}\right) - \frac{1}{3} = \frac{1}{q_0} - i\left(\frac{1}{p} - \frac{1}{3}\right) = \frac{1}{2} - i\left(\frac{1}{p} - \frac{1}{3}\right) < 0,\ \text{if } i \text{ large.}$$

Consequently, $Du \in L^q,\ 1 \le q < \infty$, and

$$\nabla\left(a\rho^\gamma + \alpha\rho^p\right) = \cdots \text{ by } (2)_2 \Rightarrow \nabla\left(a\rho^\gamma + \alpha\rho^p\right) \in L^q,\ 1 \le q < \infty.$$

□

Remark 2. *One may use many variants for the approximation of the stationary problem* (1)*, other than* (2)*, or those in* [2]*.*

Remark 3. *As we know, for* (1)*,*

1. *when $M = 0$, there exists an unique solution u of (1);*
2. *however, for $M > 0$ small, we do not have uniqueness of solutions of (1), see [1,* Remark 6.16, Page 117].

Thus, the existence result for small $M > 0$ could not be obtained by invoking (variants of) implicit function theorem (to yield an unique branch of solutions).

3. **A technical Lemma.**

Lemma 4. *Let*

1. $0 \leq \rho \in L^p(\Omega)$, $1 \leq p \leq \infty$;
2. $u \in W^{1,q}(\Omega)$, $1 \leq q \leq \infty$ *with* $u \cdot n = 0$ *on* $\partial\Omega$;
3. $\dfrac{1}{p} + \dfrac{1}{q} \leq 1$; *and*
4. *div* $(\rho u) = 0$ *in* Ω.

Then

$$\|\varphi(\rho)\|_r \leq \left\|div\ u - \varphi(\rho)\right\|_r,\ \forall\ \varphi \in C([0,\infty)). \tag{4}$$

Proof. We just prove (4) formally, with the verification being direct consequence of regularizations.

$$\begin{aligned}
&\text{div } (\rho u) = 0\\
\Rightarrow\ &\text{div } [\beta(\rho)u] = u\cdot\nabla\beta(\rho) + \beta(\rho)\text{div } u = u\nabla\beta(\rho) + \frac{\beta(\rho)}{\rho}[-u\cdot\nabla\rho] = \left[\beta'(\rho) - \frac{\beta(\rho)}{\rho}\right]u\cdot\nabla\rho\\
\Rightarrow\ &u\cdot\nabla\varphi(\rho) = \text{div } [\beta(\rho)u] \text{ for } \varphi'(\rho) = \beta'(\rho) - \frac{\beta(\rho)}{\rho}\\
&\left(t\beta'(t) - \beta(t) = t\varphi'(t) \Rightarrow \left[\tilde\beta(s) = \beta(e^s)\right]\tilde\beta'(s) - \tilde\beta(s) = e^s\varphi'(e^s)\right)\\
\Rightarrow\ &0 = \int_\Omega \text{div } [\beta(\rho)u] = \int_\Omega u\cdot\varphi(\rho) = -\int_\Omega \varphi(\rho)\text{div } u\\
\Rightarrow\ &\int_\Omega |\varphi(\rho)|^p = \int_\Omega [\varphi(\rho) - \text{div } u]\,|\varphi(\rho)|^{p-2}\varphi(\rho) \leq \|\varphi(\rho) - \text{div } u\|_p\,\|\varphi(\rho)\|_p^{p-1}\\
\Rightarrow\ &\|\varphi(\rho)\|_p \leq \|\varphi(\rho) - \text{div } u\|_p.
\end{aligned}$$

□

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