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MEAN-VALUE PROPERTY OF THE HEAT EQUATION

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ABSTRACT. In this paper, we detailed the proof of the mean-value theorem for the heat equation, see [1] for example.

Let $U \subset \mathbb{R}^n$ be open and bounded, and $T > 0$. We give

Definition 1. *1. The* **parabolic cylinder** *is the* **parabolic interior** *of* $\bar{U} \times [0, T]$*:*

$$U_T \equiv U \times (0, T].$$

2. The **parabolic boundary** *of* U_T *is*

$$\Gamma_T \equiv \bar{U}_T - U_T,$$

which comprises the bottom and vertical sides of $U \times [0, T]$*, but not the top.*

In this parabolic cylinder U_T, we want to derive a kind of analogue to the mean-value property for harmonic function. For this purpose, we introduce

Definition 2. *The* **heat ball** $E(x, t; r)(r > 0)$ *at* $(x, t) \in \mathbb{R}^{n+1}$ *is*

$$E(x, t; r) = \left\{ (y, s) \in \mathbb{R}^{n+1};\ \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}.$$

Remark 3. *1. The heat ball is a region in space-time, the boundary of which is a level set of* $\Phi(x - y, t - s)$*.*

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2. *Written explicitly, we have*

$$\frac{1}{[4\pi(t-s)]^{n/2}}e^{-\frac{|x-y|^2}{4(t-s)}} = \Phi(x-y,t-s) \geq \frac{1}{r^n},$$

$$r^n e^{-\frac{|x-y|^2}{4(t-s)}} \geq [4\pi(t-s)]^{n/2}.$$

Applying the logarithmical function, we obtain

$$n\ln r - \frac{|x-y|^2}{4(t-s)} \geq \frac{n}{2}\ln[4\pi(t-s)],$$

$$|x-y|^2 \leq 2n(t-s)\ln\frac{r^2}{4\pi(t-s)}.$$

One then verifies easily that RHS of the above inequality equal 0 *if*

$$s = t - \frac{r^2}{4\pi} \text{ or } s = t.$$

This echoes the notion of heat ball, a region in space-time, with the scale in t is twice that in x.

3. *By the above calculations, we find that the function*

$$\psi \equiv -\frac{n}{2}\ln[4\pi(t-s)] - \frac{|x-y|^2}{4(t-s)} + n\ln r, \tag{1}$$

vanishes on $\partial E(x,t;r)$, which is helpful in integration by parts formula, as we shall in later on. Notice also that

$$\psi_y = -\frac{y}{2(t-s)}, \tag{2}$$

$$\psi_s = \frac{n}{2}\frac{s}{t-s} - \frac{|x-y|^2}{4(t-s)^2}. \tag{3}$$

Now, we state and prove our mean-value theorem for the heat equation as

Theorem 4. *(A mean-value property for the heat equation). Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s)\frac{|y|^2}{s^2}\,dyds, \tag{4}$$

for each $E(x,t;r) \subset U_T$.

Proof. 1. An useful identity:

$$\iint_{E(1)} \frac{|y|^2}{s^2}\,dyds = 4, \tag{5}$$

where $E(1) = E(0,0;1)$.

Indeed,

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2}\,dyds &= \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2}ds \int_{|y|^2 \le -2ns\ln\frac{1}{-4\pi s}} |y|^2\,dy \\
&= \int_{-\frac{1}{4\pi}}^{0} \frac{ds}{s} \int_0^{[-2ns\ln\frac{1}{-4\pi s}]^{1/2}} n\alpha(n) r^{n-1+2}\,dr \\
&= \frac{n\alpha(n)}{n+2} \int_{-\frac{1}{4\pi}}^{0} \frac{1}{(-s)^2}\left[2\pi(-s)\ln\frac{1}{4\pi(-s)}\right]^{\frac{n+2}{2}} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \int_0^{\frac{1}{4\pi}} s^{\frac{n-2}{2}} \left(\ln\frac{1}{4\pi s}\right)^{\frac{n+2}{2}} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \int_\infty^0 \left(\frac{1}{4\pi}e^{-s}\right)^{\frac{n-2}{2}} \cdot s^{\frac{n+2}{2}} \cdot \left(-\frac{1}{4\pi}e^{-s}\right) ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \cdot \frac{1}{(4\pi)^{n/2}} \int_0^\infty s^{\frac{n+4}{2}-1} e^{-\frac{n}{2}s}\,ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \cdot \frac{1}{(4\pi)^{n/2}} \int_0^\infty \left(\frac{2}{n}\right)^{\frac{n+4}{2}} t^{\frac{n+4}{2}-1} e^{-t}\,dt \\
&= \frac{8}{(n+2)\pi^{n/2}} \cdot \Gamma\left(\frac{n}{2}+2\right) \\
&= \frac{8}{(n+2)\pi^{n/2}} \cdot \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot \left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right) \\
&= 4.
\end{aligned}$$

2. We now prove (4). Without loss of generality, we may assume that $(x,t)=(0,0)$. Write $E(r)=E(0,0;r)$ and set

$$\begin{aligned}\varphi(r) &\equiv \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}dyds\\ &= \iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}dyds.\end{aligned}$$

Then

$$\begin{aligned}\varphi'(r) &= \iint_{E(1)}\left[y\cdot D_yu\frac{|y|^2}{s^2}+2rD_su\frac{|y|^2}{s}\right]dyds\\ &= \frac{1}{r^{n+1}}\iint_{E(r)}\left[y\cdot D_yu\frac{|y|^2}{s^2}+2D_su\frac{|y|^2}{s}\right]dyds\\ &\equiv A+B.\end{aligned}$$

Next, we calculate B as

$$\begin{aligned}B &= \frac{1}{r^{n+1}}\iint_{E(r)} 2D_su\frac{|y|^2}{s}dyds\\ &= \frac{4}{r^{n+1}}\iint_{E(r)} D_suD_y\varphi\cdot ydyds \ \ ((2))\\ &= -\frac{4}{r^{n+1}}\int_{E(r)} y\cdot D_sD_yu\varphi dyds-\frac{4n}{r^{n+1}}\iint_{E(r)} D_su\varphi dyds\\ &\quad \left(\text{integration by part w.r.t. } y\right)\\ &= \frac{4}{r^{n+1}}\iint_{E(r)}\left\{y\cdot D_yu\left[-\frac{n}{2s}-\frac{|y|^2}{4s^2}\right]\right\}dyds\\ &\quad -\frac{4n}{r^{n+1}}\iint_{E(r)} D_su\varphi dyds\ \left(\text{integration by part w.r.t. } s \text{ and } (3)\right)\\ &= -A+\frac{4}{r^{n+1}}\iint_{E(r)}\left[-\frac{n}{2s}y\cdot D_yu-n\Delta_yu\varphi\right]dyds\ \left(D_su-\Delta_yu=0\right)\\ &= -A\left(\text{integration by part w.r.t. } y \text{ and } (2)\right).\end{aligned}$$

Hence,

$$\varphi(r) = \lim_{t\to 0_+}\varphi(t)=\lim_{t\to 0_+}\iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}dyds$$

$$= \iint_{E(1)} u(0,0)\frac{|y|^2}{s^2}dyds = 4u(0,0).$$

The proof of the mean-value property of the heat equation is thus completed. □

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