3_19[uia,math,sysu,china]设 A 是环, M 是 $A-$模,用 $Supp(M)$ 表 A 中使得 $M_{\mathfrak{p}} \neq 0$ 的素理想 $\mathfrak{p}$ 的集合,称 $Supp(M)$ 为 M 的支集(support).证明下述结果:

i) $M \neq 0 \Leftrightarrow Supp(M) \neq \varnothing$.

ii) $V(\mathfrak{a}) = Supp(A/\mathfrak{a})$.

iii)如 $0 \to M' \to M \to M'' \to 0$ 是正合序列,那么

$$Supp(M) = Supp(M') \cup Supp(M'')$$

iv)如 $M = \sum M_i$,那么 $Supp(M) = \cup Supp(M_i)$.

v)若 M 有限生成,那么 $Supp(M) = V(Ann(M))$ (因此是 $Spec(A)$ 的闭子集)

vi)若 M, N 有限生成,那么 $Supp(M \otimes N) = Supp(M) \cap Supp(N)$.

vii)若 M 有限生成, $\mathfrak{a}$ 是 A 的理想,那么 $Supp(M/\mathfrak{a}M) = V(\mathfrak{a} + Ann(M))$.

viii)若 $f: A \to B$ 是环同态, M 是有限生成 $A-$模,那么

$$Supp(B \otimes_A M) = f^{*-1}(Supp(M))$$

证明 i)因

$$[M = 0] \Leftrightarrow [\forall \mathfrak{p} \in Spec(A), M_{\mathfrak{p}} = 0]$$

有

$$\begin{aligned}
[M \neq 0] &\Leftrightarrow [\exists \mathfrak{p} \in Spec(A), s.t.\ M_{\mathfrak{p}} \neq 0] \\
&\Leftrightarrow [\mathfrak{p} \in Supp(M)] \\
&\Leftrightarrow [Supp(M) \neq \varnothing]
\end{aligned}$$

ii) $\forall \mathfrak{p} \in Spec(A)$,

$$\begin{aligned}
&[\mathfrak{p} \notin Supp(A/\mathfrak{a})] \Leftrightarrow [(A/\mathfrak{a})_{\mathfrak{p}} = 0] \\
&\Leftrightarrow \left[\forall x \in A, \forall s \in A \setminus \mathfrak{p}, \frac{\overline{x}}{s} = 0\right] \\
&\Leftrightarrow [\forall x \in A, \exists t \in A \setminus \mathfrak{p}, s.t.\ tx \in \mathfrak{a}] \\
&\Leftrightarrow [\exists t \in A \setminus \mathfrak{p}, s.t.\ t \in \mathfrak{a}] \{\Rightarrow \text{取} x = 1; \Leftarrow \mathfrak{a}\text{为理想}\} \\
&\Leftrightarrow [A \setminus \mathfrak{p} \not\subset A \setminus \mathfrak{a}] \\
&\Leftrightarrow [a \not\subset \mathfrak{p}] \\
&\Leftrightarrow [\mathfrak{p} \notin V(\mathfrak{a})]
\end{aligned}$$

iii)由

$$0 \to M' \to M \to M'' \to 0$$

正合有

$$0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$$

正合,对 $\forall \mathfrak{p} \in Spec(A)$.而

$$[M_{\mathfrak{p}} = 0] \Leftrightarrow [M'_{\mathfrak{p}} = 0 = M''_{\mathfrak{p}}]$$

于是

$$\left[\mathfrak{p}\in Supp(M)\right]\Leftrightarrow\left[M_{\mathfrak{p}}\neq 0\right]$$
$$\Leftrightarrow\left[M_{\mathfrak{p}}^{'}\neq 0 \text{ 或 } M_{\mathfrak{p}}^{''}\neq 0\right]$$
$$\Leftrightarrow\left[\mathfrak{p}\in Supp(M')\cup Supp(M'')\right]$$

iv)

$$\left[\mathfrak{p}\in \cup Supp(M_i)\right]\Leftrightarrow\left[\exists i,\ s.t.\ (M_i)_{\mathfrak{p}}\neq 0\right]$$
$$\Leftrightarrow\left[\exists i,\ s.t.\ \exists x\in M_i,\ s\in A\setminus\mathfrak{p},\ s.t.\ \frac{x}{s}\neq 0\right]$$
$$\Leftrightarrow\left[\exists x\in M,\ s\in A\setminus\mathfrak{p},\ s.t.\ \frac{x}{s}\neq 0\right]\left\{M=\sum M_i\right\}$$
$$\Leftrightarrow\left[M_{\mathfrak{p}}\neq 0\right]$$
$$\Leftrightarrow\left[\mathfrak{p}\in Supp(M)\right]$$

v)设 M 由 $\{x_i\}_{i=1}^{n}$ 生成,则

$$\left[\mathfrak{p}\notin Supp(M)\right]\Leftrightarrow\left[M_{\mathfrak{p}}=0\right]$$
$$\Leftrightarrow\left[\forall m\in M,\ \forall s\in A\setminus\mathfrak{p},\ \frac{x}{s}=0\right]$$
$$\Leftrightarrow\left[\forall m\in M,\ \exists t\in A\setminus\mathfrak{p},\ s.t.\ tx=0\right]$$
$$\Leftrightarrow\left[\forall i,\ \exists t_i\in A\setminus\mathfrak{p},\ s.t.\ t_ix_i=0\right]$$
$$\Leftrightarrow\left[\exists t\in A\setminus\mathfrak{p},\ s.t.\ tM=0\right]\left\{\Rightarrow \text{取} t=\prod_{i=1}^{n}t_i;\ \Leftarrow \text{取} t_i=t,\forall i\right\}$$
$$\Leftrightarrow\left[\exists t\in A\setminus\mathfrak{p},\ s.t.\ t\in Ann(M)\right]$$
$$\Leftrightarrow\left[A\setminus\mathfrak{p}\not\subset A\setminus Ann(M)\right]$$
$$\Leftrightarrow\left[Ann(M)\not\subset\mathfrak{p}\right]$$
$$\Leftrightarrow\left[\mathfrak{p}\notin V(Ann(M))\right]$$

vi)

$$\left[\mathfrak{p}\notin Supp(M\otimes N)\right]\Leftrightarrow\left[(M\otimes_A N)_{\mathfrak{p}}=0\right]$$
$$\Leftrightarrow\left[M_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}N_{\mathfrak{p}}=0\right]\left\{\text{命题3.7. P52}\right\}$$
$$\Leftrightarrow\left[M_{\mathfrak{p}}=0 \text{ 或 } N_{\mathfrak{p}}=0\right]\left\{\text{习题2.3. P42}\right\}$$
$$\Leftrightarrow\left[\mathfrak{p}\notin Supp(M)\cap Supp(N)\right]$$

vii)

$$\left[\mathfrak{p}\notin Supp(M/\mathfrak{a}M)\right]\Leftrightarrow\left[(M/\mathfrak{a}M)_{\mathfrak{p}}=0\right]$$

$$\Leftrightarrow\left[\forall m\in M,\forall s\in A\setminus\mathfrak{p},\frac{\overline{m}}{s}=0\right]$$

$$\Leftrightarrow\left[\forall m\in M,\exists t\in A\setminus\mathfrak{p},s.t.\ m\in\mathfrak{a}m\right]$$

$$\Leftrightarrow\left[\exists t\in A\setminus\mathfrak{p},s.t.\ tM\subset\mathfrak{a}M\right]\{\text{注意到}M\text{是有限生成的}\}$$

$$\left|\begin{array}{l}\Rightarrow\left[\exists t\in A\setminus\mathfrak{p},s.t.\ t\in r(\mathfrak{a}+Ann(M))\right]\\ \quad\left\{\begin{array}{l}\text{由命题2.4. P28},t^n+a_1t^{n-1}+\cdots+a_n=0,\text{而}\\ t^n=-a_1t^{n-1}-\cdots-a_n+Ann(M)\subset\mathfrak{a}+Ann(M)\end{array}\right\}\\ \Leftarrow\left[\exists t\in A\setminus\mathfrak{p},s.t.\ t\in\mathfrak{a}+Ann(M)\right]\end{array}\right.$$

$$\left|\begin{array}{l}\Leftrightarrow\left[A\setminus\mathfrak{p}\not\subset A\setminus r(\mathfrak{a}+Ann(M))\right]\\ \Leftrightarrow\left[A\setminus\mathfrak{p}\not\subset A\setminus(\mathfrak{a}+Ann(M))\right]\end{array}\right.$$

$$\left|\begin{array}{l}\Leftrightarrow\left[r(\mathfrak{a}+Ann(M))\not\subset\mathfrak{p}\right]\\ \Leftrightarrow\left[\mathfrak{a}+Ann(M)\not\subset\mathfrak{p}\right]\end{array}\right.$$

$$\Leftrightarrow\left[\mathfrak{a}+Ann(M)\not\subset\mathfrak{p}\right]\{\text{通过取根}\}$$

$$\Leftrightarrow\left[\mathfrak{p}\notin V(\mathfrak{a}+Ann(M))\right]$$

viii)这是错误的. 构造反例如下: $f:A=\mathbb{Z}\to B=\mathbb{Z}/6\mathbb{Z}$, $M=\mathbb{Z}/7\mathbb{Z}$ 作为 $A=\mathbb{Z}$ 模是有限生成的.

- $B\otimes_A M=(\mathbb{Z}/6\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/7\mathbb{Z})=0$ [习题 2.1. P41],而又由习题 3.19 i) 有

$$Supp(B\otimes_A M)=\varnothing$$

- 由

$$Supp(M)=\left\{\mathfrak{p}=(p);p=0\text{或为素数},(\mathbb{Z}/7\mathbb{Z})_{\mathfrak{p}}=0\right\}$$

$$=\left\{\mathfrak{p};\forall\overline{i}\in\mathbb{Z}/7\mathbb{Z},\forall s\in\mathbb{Z}\setminus\mathfrak{p},\frac{\overline{i}}{s}=0\right\}$$

$$=\left\{\mathfrak{p};\forall\overline{i}\in\mathbb{Z}/7\mathbb{Z},\exists t\in\mathbb{Z}\setminus\mathfrak{p},s.t.\ t\overline{i}=0\right\}$$

$$=\left\{\mathfrak{p};\exists t\in\mathbb{Z}\setminus\mathfrak{p},s.t.\ t\mathbb{Z}\subset 7\mathbb{Z}\right\}\{\mathbb{Z}/7\mathbb{Z}\text{为有限生成的}\}$$

$$=\left\{\mathfrak{p};\exists t\in\mathbb{Z}\setminus\mathfrak{p},s.t.\ 7\mid t\right\}$$

$$=\{(p),p\neq 7,p\text{为0或素数}\}$$

有

$$f^{*-1}\left(Supp(M)\right) = f^{*-1}\left\{(p), p \neq 7, p\text{为}0\text{或素数}\right\}$$

$$\supset f^{*-1}\left\{(2),(3)\right\}$$

$$= \left\{\overline{0},\overline{2},\overline{4}\right\},\left\{\overline{0},\overline{3}\right\}$$

[关于 $\mathbb{Z}/6\mathbb{Z}$ 中素理想,参考习题 3.5 的解答]