Riemanian Geometry

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1.Let M be a Riemannian Manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping

$$exp_p : B_\varepsilon(0) \subset T_pM \to B_\varepsilon(p)$$

is an isometry, where $B_\varepsilon(p)$ is a normal ball at p.

2.Let $\tilde{M}$ be a covering space of a Riemanian Manifold M. Show that it is possible to give $\tilde{M}$ a Riemannian structure such that the covering map $\pi : \tilde{M} \to M$ is a local isometry (this metric is called the covering metric). Show that $\tilde{M}$ is complete in the covering metric iff M is complete.

3.If a complete simply connected Riemanian Manifold M has a pole, then M is diffeomorphic to R^n, $n = dim M$.

4. Introduce a complete Riemannian metric on $\mathcal{R}^2$. Prove that

$$\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x,y)\right)\leq 0$$

where $(x,y)\in\mathcal{R}^2$ and $K(x,y)$ is the Gaussian curvature of the given metric at (x,y).

5. Let $\gamma:[0,a]\to M$ be a geodesic segment on M such that $\gamma(a)$ is not conjugate to $\gamma(0)$. Then γ has no conjugate points on $(0,a)$ iff for all proper variations of γ

$$\exists\delta>0,\ s.t.\ 0<|s|<\delta\Rightarrow E(s)>E(0)$$

In particular, if γ is minimizing, γ has no conjugate points on $(0,a)$.