Comment on "Conjectures on exact solution of three-dimensional (3D) simple orthorhombic Ising lattices"*

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Abstract

It is shown that a recent article by Z.-D. Zhang is in error and violates well-known theorems.

After receiving an electronic reprint of Zhang's recent paper [1], I have had an email exchange with the author pointing out a number of errors in the paper, which unfortunately invalidate all its main results. As now also a follow-up preprint [2] has appeared using Zhang's erroneous results, I feel compelled to write down some of my criticism.

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One of the main results of [1] is formula (49) for the partition function per site on page 5325 [(3.37) on page 26][1], which has three parameters given in the appendix. On page 5399 [page 137] one finds eqs. (A.1), (A.2) and the following text, where these three parameters are expressed as $w_x = 1$, $w_y = w_z$ equal to an expression with the coefficients b_0 through b_{10} fitted such that the high-temperature series is recovered. Therefore, this expression for the free energy contains no more information than the known coefficients of the high-temperature series used.

On page 5400 [page 139] Zhang insists that $w_y = w_z \equiv 0$ as soon as the temperature is finite. This is discussed further in eqs. (A.11)–(A.13) on pages 5405–5406 [pages 145–146], with κ the usual high-temperature variable $\tanh K$. There is a marked difference between the "high-temperature limit" (A.11) and eq. (A.13) for more general temperature, as the author chooses $w_x = 1$, and $w_y = w_z = 0$, as soon as the temperature is finite, which is highly inconsistent with the earlier fit.

Indeed, the procedure is clearly wrong as the convergence of the high-temperature series has been rigorously proved in the 1960s [3, 4] and this proof has been quoted in many textbooks [5, 6, 7]. This proof is based on the proof of Gallavotti and Miracle-Solé [3] of the convergence of the fugacity expansion by a use of the Kirkwood–Salzburg equations for the lattice-gas, which is equivalent to the Ising model. Another theorem of Lebowitz and Penrose [4] is then used to establish a finite radius of convergence for the correlation functions and the free energy expressed as series in $1/T$. They are even real analytic up to a critical point [4, 6]. Paper [1] therefore violates well-established theorems. The statements on page 5376 [page 102] are, therefore, manifestly wrong.

Another criticism concerns the result for the spontaneous magnetization given in eqs. (102) and (103) on page 5342 [(4.28) and (4.29) on page 50]. This can be expanded as $I = 1 - 6x^8 + \ldots$, with $x = \exp(-2K)$, with $K = J/k_{\mathrm{B}}T$. However, in Table 2 on page 5380 [page 154] one finds $I = 1 - 2x^6 + \ldots$, taken from the well-known low-temperature series in the literature. Zhang's result is analytic in the low-temperature variable x, up to his critical point and it also gives the exact value $I = 1$ at $T = 0$. It has a finite radius of convergence expanded as a series in x. Therefore, it must agree with the well-known series result in Table 2, which it does not.

It has also been established that in the ferromagnetic Ising model the

[1] References to the arXiv preprint are given within square brackets.

thermodynamic bulk limit converges to a unique state, apart from $H = 0$, $T < T_c$ where the state can be any convex combination of the states obtained by the infinite-volume limits with all boundary spins up or all down, see [7] and references quoted. One can then study an infinite hierachy of a discrete version of the Schwinger–Dyson equations connecting the correlation functions with an odd number of spins in the thermodynamic limit with all spins up on the boundary. This way one can easily and rigorously establish the start of the low-temperature series for the spontaneous magnetization, in agreement with the old results in the literature. Hence, because of the discrepancy, Zhang's result is manifestly wrong.

More can be said about the correlation functions, susceptibility, and critical exponents in sections 5, 6, and 7. Again, all the main results are in error. I will not go into more detail as this should already be clear from the arguments above.

References

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[5] D. Ruelle, *Statistical Mechanics, Rigorous Results*, Benjamin, NY, 1969.

[6] R.B. Griffiths, *Phase Transitions and Critical Phenomena*, Vol. 1, edited by C. Domb and M.S. Green, Academic Press, London, 1972, chapter 2, sections III and IV D.

[7] S. Miracle-Solé, Theorems on phase transitions with a treatment for the Ising model, in *Lecture Notes in Physics*, vol. 54, Springer, 1976, pp. 189–214.