Weak Limits

Theorem Let $\{g^n\}$ be bounded in $L^\infty(I)$, with $I \subset R^1$, $|I| < \infty$.

Then

- $g^n \to g$ in $L^p(I)$ weakly for some $g \in L^p(I)$, $\forall 1 < p < \infty$.
- Furthermore, $g \in L^\infty(I)$.

Proof The weak convergence follows from the reflexivity of L^p $(1 < p < \infty)$.

We then focus our attention to the boundedness of $\|g\|_{L^p}$.

Let

$$\sup_{n\geq 1}\left\|g^n\right\| = M < \infty,$$

and

$$g^n \to g \text{ weakly in } L^p.$$

Then Mazur tells us that

$$h_k = \sum_i \lambda_i^{(k)} g^{n_i^{(k)}} \to g \text{ strongly in } L^p.$$

Thus

$$\|g\|_{L^p} \leq M.$$