Weak Closure=Strong Closure

Theorem Let X be a Banach space, Y be a closed linear subspace of X. If

$$oldsymbol{x}_n \in oldsymbol{Y} \longrightarrow oldsymbol{x} \in oldsymbol{X}$$
 ,

then

 $\mathbf{x} \in \mathbf{Y}$.

Proof Suppose not, then Hahn-Banach theorem implies

$$\exists \ oldsymbol{x}^{\star} \ \in \ oldsymbol{X}^{\star}, \ oldsymbol{s.t.} \ oldsymbol{x}^{\star} \ ert_{\mathbf{y}} = \ \mathbf{0} \ \mathbf{\&} \ oldsymbol{x}^{\star} \ (oldsymbol{x}) = \ \mathbf{0} \ .$$

While by hypothesis,

$$\mathbf{0} = \left\langle \mathbf{x}^{\star}, \, \mathbf{x}_{n} \right\rangle \,
ightarrow \, \left\langle \mathbf{x}^{\star}, \, \mathbf{x} \right
angle \, = \, \mathbf{1} \, ,$$

a contradiction which completed the proof.

Example Let
$$ho^{n} \in \texttt{L}^{\infty}\left(\texttt{0, T; L}^{\gamma}\left(\texttt{R}^{n}
ight)
ight)$$
 $(\texttt{1} < \gamma < \infty)$ be

bouned, then

• Since L^{γ} is reflexive, $ho^n \xrightarrow{w}
ho$ for some $ho \in L^{\gamma}\left(0, T; L^{\gamma}\left(\mathbb{R}^n
ight)
ight);$

• By theorem just above, we have $ho~\in~ L^{\infty}\left(0,\,T;\,L^{\gamma}\left({ extsf{R}}^{n}
ight)
ight).$

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