Weak Closure=Strong Closure

Theorem Let X be a Banach space, Y be a closed linear subspace of X. If

$$x_n \in Y \xrightarrow{w} x \in X,$$

then

$$x \in Y.$$

Proof Suppose not, then Hahn-Banach theorem implies

$$\exists\ x^* \in X^*,\ s.t.\ x^*|_Y = 0\ \&\ x^*(x) = 0.$$

While by hypothesis,

$$0 = \langle x^*, x_n \rangle \to \langle x^*, x \rangle = 1,$$

a contradiction which completed the proof.

Example Let $\rho^n \in L^\infty\left(0, T; L^\gamma\left(R^n\right)\right)$ $(1 < \gamma < \infty)$ be bouned, then

- Since L^γ is reflexive, $\rho^n \xrightarrow{w} \rho$ for some $\rho \in L^\gamma\left(0, T; L^\gamma\left(R^n\right)\right)$;
- By theorem just above, we have $\rho \in L^\infty\left(0, T; L^\gamma\left(R^n\right)\right)$.

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