# Remarks on one component regularity for the Navier-Stokes equations III

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Oct. 12th, 2009

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## Abstract

We establish sufficient conditions for the regularity of solutions of the Navier-Stokes system based on one component of the velocity. It is proved that if $u_3 \in L^{s,r}$ with

$$\frac{2}{s} + \frac{3}{r} \leq \frac{3}{4}$$

and $4 < r \leq \infty$, then the solution is regular.

## 1. Introduction and the main result

We continue our study in [1] of the one component regularity problem for Leray-Hopf weak solutions $(u, p)$ of the three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} \partial_t u + u.\nabla u - \Delta u + \nabla p = 0; \\ div\, u = 0. \end{cases}$$

Indeed, the main result is as follows.

**Theorem 1.1** Let $u$ be a Leray-Hopf weak solution of (NS) with data $u_0 \in H^1(R^3)$, and $u_3 \in L^{s,r}$ with

$$\frac{2}{s} + \frac{3}{r} \le \frac{3}{4}, \qquad 4 < r \le \infty$$

Then $u$ is actually regular.

The proof is similar to that in [1], so we just estimate the key term $L_1$.

## 2. Proof of the main theorem

As in [1], we set $\theta_0 = \frac{3}{4}$, and we have

$$J \le C\varepsilon L^{3/4} + C.$$

Now we estimate $L_1$ more carefully.

$$L_1 \le C\iint |\nabla\nabla_h u|\,|u|\,|\partial_3 u|$$

$$\le C\int |\nabla\nabla_h u|_2\,|u|_6\,|\partial_3 u|_2^{1/2}\,|\partial_3 u|_6^{1/2}$$

$$\left\{ Holder\ inequality\ with\ \frac{1}{2} + \frac{1}{6} + \frac{1/2}{2} + \frac{1/2}{6} = 1 \right\}$$

$$\le C\int |\nabla\nabla_h u|_2 \left[ |\nabla_h u|_2^{2/3}\,|\partial_3 u|_2^{1/3} \right] |\partial_3 u|_2^{1/2} \left[ |\nabla\nabla_h u|_2^{1/3}\,|\nabla\partial_3 u|_2^{1/6} \right]$$

$$\left\{ Multiplicative\ Sobolev\ imbedding,\ |u|_6 \le |\nabla_h u|_2^{2/3}\,|\partial_3 u|_2^{1/3} \right\}$$

$$\le C\int |\nabla\nabla_h u|_2^{4/3}\,|\nabla_h u|_2^{2/3}\,|\partial_3 u|_2^{1/3}\,|\partial_3 u|_2^{1/2}\,|\nabla\partial_3 u|_2^{1/6}$$

$$\le C\,|\nabla\nabla_h|_{2,2}^{4/3}\,|\nabla_h u|_{\infty,2}^{2/3}\,|\partial_3 u|_{\infty,2}^{1/3}\,|\partial_3 u|_{2,2}^{1/2}\,|\nabla\partial_3 u|_{2,2}^{1/6}$$

$$\left\{ Holder\ inequality\ with\ \frac{4/3}{2} + \frac{1/2}{2} + \frac{1/6}{2} = 1 \right\}$$

$$\leq C\varepsilon J^2 L^{1/2}$$

$$\leq C\varepsilon L^{2\cdot 3/4+1/2} + C$$

$$\left\{ J \leq C\varepsilon L^{4/3} + C \right\}$$

$$\leq C\varepsilon L^2 + C$$

For sufficiently small $\varepsilon$, we obtain $L \leq C$, thus $J \leq C$ also. The proof is complete.

## 3. Acknowledgement

The author would like to express his sincere gratitude to Professor Zhou from ZNU and Professor Kukavica from USC. This paper comes out of their inspiring papers.

## 4. A Note

Why I choose the font---Courier New? This is because I've read Leon Simon's << Lectures on Geometric Measure theory >>, and this is just the font he use. I feel awful first, but then like it very much as I suffer through……

Worse than Zhou's, the result is still an exercise.

## 5. Reference

[1] Z. Zhang, Remarks on one component regularity for the Navier-Stokes equations, an exercise.