Remarks on one component regularity for the Navier-Stokes equations

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Abstract

We establish sufficient conditions for the regularity of solutions of the Navier-Stokes system based on one component of the velocity. It is proved that if $u_3 \in L_t^s L_x^r$ with

$$\frac{2}{s}+\frac{3}{r}\leq\frac{15}{22}$$

and $22/5 < r \leq \infty$, then the solution is regular.

Key words: Navier-Stokes equations, weak solutions, strong solutions, regularity

1 Introduction and Main Result

We consider the following Cauchy problem for the incompressible Naiver-Stokes equations in $\mathbb{R}^3 \times (0,T)$:

$$\begin{cases} \frac{\partial u}{\partial t}+u\cdot\nabla u-\triangle u+\nabla p=0, \\ \text{div } u=0, \\ u(x,0)=u_0(x). \end{cases} \tag{1.1}$$

where $u=(u_1(x,t),u_2(x,t),u_3(x,t))$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with div $u_0=0$ in distributional sense is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long histroy(see [2, 9]). In the fundamental work [6] and [3], Leray and Hopf proved the existence of its weak solutions $u(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ for given initial data $u_0(x) \in L^2(\mathbb{R}^3)$. But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [7], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli-Kohn and Nirenberg in [1]. Further results can be found in [10] and references therein.

On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin[8] proved that if u is a Leray-Hopf weak solution belonging to $L^{s,r} = L^s(0,T;L^r(\mathbb{R}^3))$ with $2/s + 3/r \leq 1, 2 < s < \infty, 3 < r < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$. From then on, there are many criterion results added on $u, \nabla u$, one or two components of these, the pressure(see [11] and reference therein). Here $L^{s,r}$ is defined by

$$|u|_{s,r} = ||u||_{L^{s,r}} = \begin{cases} \left(\int_0^T |u|_r^s(\tau)d\tau\right)^{1/s}, & \text{if } 1 \leq s < \infty, \\ \operatorname{esssup}_{0\leq\tau\leq T}|u|_r(\tau), & \text{if } s = \infty. \end{cases}$$

where

$$|u|_r(\tau) = ||u||_{L^r}(\tau) = \begin{cases} \left(\int_{\mathbb{R}^3} |u(x,\tau)|^r dx\right)^{1/r}, & \text{if } 1 \leq r < \infty, \\ \operatorname{esssup}_{x\in\mathbb{R}^3}|u(x,\tau)|, & \text{if } r = \infty. \end{cases}$$

The point is that $|u_\lambda|_{s,r} = |u|_{s,r}$ holds for all $\lambda > 0$ if and only if $2/s + 3/r = 1$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ and if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$ (See the dimensions table in [1]). Usually we say that the norm $|u|_{s,r}$ has the scaling dimension zero for $2/s + 3/r = 1$.

In this paper, we shall improve these and other known one-component regularity result. Our main result is

Theorem 1.1. *Let u be a Leary-Hopf weak solution of* (1.1) *with data $u_0 \in H^1(\mathbb{R}^3)$, and $u_3 \in L^{s,r}$ with*

$$\frac{2}{s} + \frac{3}{r} \leq \frac{15}{22}, \quad \frac{22}{5} < r \leq \infty$$

then u is regular in $[0,T)$.

Remark 1.1. *It improves the result in [4] which says that if $u_3 \in L^{s,r}$ with*

$$\frac{2}{s} + \frac{3}{r} \leq \frac{5}{8}, \quad \frac{24}{5} < r \leq \infty$$

then u is regular in $(0,T)$. However our proof is based on his method but more involved, using Hölder inequality and Young inequality freely to avoid technical computations and exhaust all possibilities.

Remark 1.2. *Since $15/22<1$, our result is not optimal in the sense of dimensional analysis. And it is remarked by Zhou in [11] that it is indeed a challenging problem to show regularity by adding Serrin's condition on only one velocity component. The key point is that one should lower the sum of powers in the convective terms from 3 to less than 2. But only the terms with u_3 (or perhaps ∇u) are possible.*

We will use the following two results.

Lemma 1.1. *[5] Assume that $u=(u_1,u_2,u_3)\in H^2(\mathbb{R}^3)$ is smooth and divergence free, then*

$$\sum_{i,j=1}^{2}\int u_i\partial_i u_j\triangle_h u_j=\frac{1}{2}\sum_{i,j=1}^{2}\int\partial_i u_j\partial_i u_j\partial_3 u_3-\int\partial_1 u_1\partial_2 u_2\partial_3 u_3+\int\partial_1 u_2\partial_2 u_1\partial_3 u_3$$

where $\triangle_h=\partial_{11}^2+\partial_{22}^2=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ is the two dimensional Laplacian.

Lemma 1.2. *[11] Assume $u\in L^{\infty,2}$ and $\nabla u\in L^{2,2}$ on $[0,T)$, then $u\in L^{a,b}$ with $2/a+3/b\geq 3/2, 2\leq b\leq 6$. Moreover,*

$$|u|_{a,b}\leq C(p,q,T)|u|_{\infty,2}^{3/b-1/2}|\nabla u|_{2,2}^{3/2-3/b}$$

In this paper, unless otherwise stated, we shall denote by C a generic constant depending only on the initial data,$|\nabla u|_2(t_1)$ (t_1 defined below), and may differ from line to line, and ε a sufficient small constant which may differ by some power. Also, we will use Hölder inequality, Young inequality freely, sometimes without explanation. The reader may keep in mind that $\{a,b\},\{c,d\},\{p,q\}$ are Hölder conjugates, $\theta,\sigma,\tau\in[0,1]$.

2 Proof of Theorem 1.1

First note that as $u_0\in H^1(\mathbb{R}^3)$, there is possibly a short time interval $(0,T^*)$ such that there exists a strong solution to (1.1) such that $u\in C^\infty((0,T^*)\times\mathbb{R}^3)$. Denote by $\mathcal{T}^*$ the supremum of of all such T^*, and assume $\mathcal{T}^*<T$. In what follows we will show that $|\nabla u|_2(t)$ remains bounded independently of $t\to\mathcal{T}^*_-$. The standard extension argument leads to contradiction.

We fix $\varepsilon>0$ sufficiently small and $t_1<\mathcal{T}^*$ such that

$$\int_{t_1}^{\mathcal{T}^*}|u_3|_r^s d\tau<\varepsilon,\quad\int_{t_1}^{\mathcal{T}^*}|\nabla u|_2^2 d\tau<\varepsilon$$

(therefore we need $s < \infty$). We take $t_2 \in (t_1, \mathcal{T}^*)$ arbitrary and our aim is to show that $|\nabla u|_2(t_2) \leq C$, here C may depend on the data, $|u_3|_{r,s}$ and $|\nabla u|_2(t_1)$, but is independent of t_2. Passing with t_2 to $\mathcal{T}^*$ we get the result.

We will work with

$$J(t_2) = |\nabla_h u|_{\infty,2} + |\nabla\nabla_h u|_{2,2}$$

$$L(t_2) = |\partial_3 u|_{\infty,2} + |\nabla\partial_3 u|_{2,2}$$

$$K(t_2) = |u_h^3|_{\infty,2}^{1/3} + |\nabla u_h^3|_{2,2}^{1/3}$$

where $\nabla_h = (\partial_1, \partial_2)$,$u_h = (u_1, u_2)$.

Here and thereafter the integral is over $(t_1, t_2) \times \mathbb{R}^3$.

As a matter of fact, we need to show that

$$J(t_2) + L(t_2) \leq const < \infty$$

uniformly in t_2.

To this aim, we shall in the subsections below, estimate the terms mentioned above. For convenience, we denote by

$$\theta_0 = \frac{15}{22}$$

and write J, K, L for $J(t_2), K(t_2), L(t_2)$ respectively.

2.1 $J \leq C\varepsilon L^{\theta_0} + C$

Multiply (1.1) by $-\triangle_h u$ and integrate over $(t_1, t_2) \times \mathbb{R}^3$, we obtain

$$\begin{aligned} & \frac{|\nabla_h u|_2^2(t_2)}{2} + |\nabla\nabla_h u|_{2,2}^2 = \frac{|\nabla_h u|_2^2(t_1)}{2} + \iint u \cdot \nabla u \cdot \triangle_h u \\ = & \frac{|\nabla_h u|_2^2(t_1)}{2} + \iint \left[\sum_{i,j=1}^{2} u_j \partial_j u_i \triangle_h u_i + \sum_{j=1}^{3} u_j \partial_j u_3 \triangle_h u_3 + \sum_{i=1}^{2} u_3 \partial_3 u_i \triangle_h u_i \right] \\ = & \frac{|\nabla_h u|_2^2(t_1)}{2} + J_1 + J_2 + J_3 \qquad (2.1) \end{aligned}$$

We will estimate the J_i's differently.

$$
\begin{aligned}
J_1 &= \iint \sum_{i,j=1}^{2} u_j \partial_j u_i \triangle_h u_i \\
&= \iint \left[\frac{1}{2} \partial_3 u_3 \partial_i u_j \partial_i u_j - \partial_3 u_3 \partial_1 u_1 \partial_2 u_2 + \partial_3 u_3 \partial_1 u_2 \partial_2 u_1 \right] \\
&= \iint \left[u_3 \partial^2_{3i} u_j \partial_i u_j + u_3 \partial^2_{31} u_1 \partial_2 u_2 + u_3 \partial_1 u_1 \partial^2_{23} u_2 \right] \\
&\quad + \iint \left[-u_3 \partial^2_{31} u_2 \partial_2 u_1 - u_3 \partial_1 u_2 \partial^2_{32} u_1 \right] \\
&\leq C |u_3|_{s,r} |\nabla_h u|_{2s/(s-2),2r/(r-2)} |\nabla \nabla_h u|_{2,2} \\
&\leq C \varepsilon J^2,
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
J_2 &= \iint \sum_{j=1}^{3} u_j \partial_j u_3 \triangle_h u_3 \\
&= \iint \sum_{j=1}^{3} \sum_{k=1}^{2} -\partial_k u_j \partial_j u_3 \partial_k u_3 \\
&= \iint \left[\partial^2_{jk} u_j u_3 \partial_k u_3 + \partial_k u_j u_3 \partial^2_{jk} u_3 \right] \\
&\leq C |u_3|_{s,r} |\nabla_h u|_{2s/(s-2),2r/(r-2)} |\nabla \nabla_h u|_{2,2} \\
&\leq C \varepsilon J^2,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
J_3 &= \iint \sum_{k=1}^{2} u_3 \partial_3 u_k \triangle_h u_k \\
&\leq C |u_3|_{s,r} |\partial_3 u|_{a,b} |\triangle_h u|_{2,2} \\
&\leq |u_3|_{s,r} |\nabla u|^{1-\theta}_{2,2} |\partial_3 u|^{\theta}_{a_1,b_1} |\triangle_h u|_{2,2} \\
&\leq C \varepsilon L^{\theta} J,
\end{aligned} \tag{2.4}
$$

where

$$
\begin{cases}
\frac{1}{s} + \frac{1}{a} = \frac{1}{2} & \frac{1}{r} + \frac{1}{b} = \frac{1}{2} \\
\frac{1}{a} = \frac{1-\theta}{2} + \frac{\theta}{a_1} & \\
\frac{1}{b} = \frac{1-\theta}{2} + \frac{\theta}{b_1} & \\
\frac{2}{a_1} + \frac{3}{b_1} \geq \frac{3}{2} & 2 \leq b_1 \leq 6
\end{cases} \tag{2.5}
$$

Thus we deduce easily that

$$
\theta \geq \theta_0. \tag{2.6}
$$

If we take the equality in (2.6), then (2.4) implies

$$J_3 \leq C\varepsilon L^{\theta_0} J \tag{2.7}$$

Combine (2.2),(2.3),(2.7) and substitute into (2.1), we obtain

$$J \leq C\varepsilon L^{\theta_0} + C \tag{2.8}$$

2.2 $K \leq C\varepsilon L^{6\theta_0/5} + C$

Multiply $(1.1)_h$ by u_h^5 and integrate over $(t_1, t_2) \times \mathbb{R}^3$, we obtain

$$\frac{1}{6}|u_h^3|_2^2(t_2) + \frac{5}{9}|\nabla u_h^3|_{2,2}^2 = 5 \iint \sum_{k=1}^{2} p u_k^4 \partial_k u_k + \frac{1}{6}|u_h^3|_2^2(t_1) \tag{2.9}$$

By the well-known identity

$$-\triangle p = \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i,$$

we can write $p = p_1 + p_2 + p_3$, where the p_i's solve

$$-\triangle p_1 = \sum_{i,j=1}^{2} \partial_i u_j \partial_j u_i$$

$$-\triangle p_2 = 2\sum_{i=1}^{2} \partial_i(u_3 \partial_3 u_i) + \partial_3(u_3 \partial_3 u_3)$$

$$-\triangle p_3 = u_3 \partial_{33}^2 u_3$$

Now (2.9) implies

$$\begin{aligned} \frac{1}{6}|u_h^3|_2^2(t_2) + \frac{5}{9}|\nabla u_h^3|_{2,2}^2 &= \sum_{l=1}^{3} 5 \iint \sum_{k=1}^{2} p_l u_k^4 \partial_k u_k + \frac{1}{6}|u_h^3|_2^2|(t_1) \\ &= \sum_{l=1}^{3} K_l + \frac{1}{6}|u_h^3|_2^2(t_1) \end{aligned} \tag{2.10}$$

We will estimate the K_l's differently. For K_1,

$$\begin{aligned}
K_1 &= 5\iint \sum_{k=1}^{2} p_1 u_k^4 \partial_k u_k \\
&\leq C|p_1|_{a,b}|u_h^4|_{c,d}|\nabla_h u|_{p,q} \\
&\leq C|\triangle p_1|_{a,1/(1/b+2/3)}|u_h|_{4c,4d}^4|\nabla_h u|_{p,q} \\
&\leq C|\nabla_h u|^2_{2a,2/(1/b+2/3)}|u_h|_{4c,4d}^4|\nabla_h u|_{p,q} \\
&\leq C|\nabla u|_{2,2}^{2(1-\theta)}|\nabla_h u|^{2\theta}_{a_1,b_1}|u|^{4(1-\sigma)}_{c_1,d_1}|u_h^3|^{4\sigma/3}_{c_2/3,d_2/3}|\nabla u|^{1-\tau}_{2,2}|\nabla_h u|^{\tau}_{p_1,q_1} \\
&\leq C\varepsilon J^{2\theta}K^{4\sigma}J^{\tau} \\
&\leq C\varepsilon J^{\tau+2\theta}K^{4\sigma} \\
&\leq C\varepsilon K^6 + C\varepsilon J^{3(\tau+2\theta)/(3-2\sigma)} \\
&\leq C\varepsilon K^6 + C\varepsilon L^{3\theta_0(\tau+2\theta)/(3-2\sigma)} + C
\end{aligned} \tag{2.11}$$

In the sequel, we use Hölder inequality, the decomposition of p, Lemma 1.2, and finally $J \leq C\varepsilon L^{\theta_0} + C$ just proved in the previous subsection. More precisely, we shall have

$$\begin{cases}
\frac{1}{a}+\frac{1}{c}+\frac{1}{p}=1 & \frac{1}{b}+\frac{1}{d}+\frac{1}{q}=1 \\
\frac{1}{2a}=\frac{1-\theta}{2}+\frac{\theta}{a_1} & \frac{1}{2}\left(\frac{1}{b}+\frac{2}{3}\right)=\frac{1-\theta}{2}+\frac{\theta}{b_1} \\
\frac{1}{4c}=\frac{1-\sigma}{c_1}+\frac{\sigma}{c_2} & \frac{1}{4d}=\frac{1-\sigma}{d_1}+\frac{\sigma}{d_2} \\
\frac{1}{p}=\frac{1-\tau}{2}+\frac{\tau}{p_1} & \frac{1}{q}=\frac{1-\tau}{2}+\frac{\tau}{q_1}
\end{cases}$$

and

$$\begin{cases}
\frac{2}{a_1}+\frac{2}{b_1}\geq\frac{3}{2} & 2\leq b_1\leq 6 \\
\frac{2}{c_1}+\frac{2}{d_1}\geq\frac{3}{2} & 2\leq d_1\leq 6 \\
\frac{2}{c_2}+\frac{2}{d_2}\geq\frac{1}{2} & 6\leq d_2\leq 18 \\
\frac{2}{p_1}+\frac{3}{q_1}\geq\frac{3}{2} & 2\leq q_1\leq 6
\end{cases}$$

It follows that

$$\begin{cases}
1-\theta+\frac{2\theta}{a_1}+\frac{4(1-\sigma)}{c_1}+\frac{4\sigma}{c_2}+\frac{1-\tau}{2}+\frac{\tau}{p_1}=1 \\
1-\theta+\frac{2\theta}{b_1}-\frac{2}{3}+\frac{4(1-\sigma)}{d_1}+\frac{4\sigma}{d_2}+\frac{1-\tau}{2}+\frac{\tau}{q_1}=1
\end{cases}$$

Hence

$$\tau+2\theta \geq \frac{13}{2}-4\sigma \tag{2.12}$$

If the equality in (2.12) holds, then $7/8 \leq \sigma \leq 1$ and

$$\frac{3\theta_0(\tau+2\theta)}{3-2\sigma} = 3\theta_0\frac{13/2-4\sigma}{3-2\sigma}$$

is an increasing function of σ, attains its minimum $36\theta_0/5$ at $\sigma = 7/8$. In this case we can take as in [4],

$$a_1 = 3, b_1 = \frac{9}{4};\ c_1 = \frac{10}{3}, d_1 = \frac{10}{3};\ c_2 = 10, d_2 = 10;\ p_1 = 3, q_1 = \frac{18}{5}$$
$$\theta = 1;\ \sigma = \frac{7}{8};\ \tau = 1$$
$$a = 6, b = \frac{9}{2};\ c = 2, d = 2;\ p = 3, q = \frac{18}{5}$$

Finally (2.11) gives

$$K_1 \leq C\varepsilon K^6 + C\varepsilon L^{36\theta_0/5} + C \tag{2.13}$$

For K_2,

$$\begin{aligned} K_2 &= 5 \iint \sum_{k=1}^{2} p_2 u_k^4 \partial_k u_k \\ &\leq C|p_2|_{a,b}|u_h^4|_{c,d}|\nabla_h u|_{p,q} \\ &\leq C|\nabla p_2|_{a,1/(1/b+1/3)}|u_h|_{4c,4d}^4|\nabla_h u|_{p,q} \\ &\leq C|u_3\partial_3 u|_{a,1/(1/b+1/3)}|u_h|_{4c,4d}^4|\nabla_h u|_{p,q} \\ &\leq C|u_3|_{s,r}|\partial_3 u|_{a_1,b_1}|u|_{c_1,d_1}^{4(1-\theta)}|u_h^3|_{c_2/3,d_2/3}^{4\theta/3}|\nabla_h u|_{p,q} \\ &\leq C\varepsilon L K^{4\theta} J \\ &\leq C\varepsilon L^{1+\theta_0} K^{4\theta} \\ &\leq C\varepsilon K^6 + C\varepsilon L^{3(1+\theta_0)/(3-2\theta)} \end{aligned} \tag{2.14}$$

In the sequel, we shall have

$$\begin{cases} \frac{1}{s} + \frac{1}{a_1} + \frac{4(1-\theta)}{c_1} + \frac{4\theta}{c_2} + \frac{1}{p} = 1 \\ \frac{1}{r} + \left(\frac{1}{b_1} - \frac{1}{3}\right) + \frac{4(1-\theta)}{d_1} + \frac{4\theta}{d_2} + \frac{1}{q} = 1 \end{cases}$$

and it is easily deduced that

$$\theta \geq \frac{3+\theta_0}{4} \tag{2.15}$$

if we invoke Lemma 1.2 to verify (2.14). However, if the equality in (2.15) holds,we have

$$K_2 \leq C\varepsilon K^6 + C\varepsilon L^{6(1+\theta_0)/(3-\theta_0)} \leq C\varepsilon K^6 + C\varepsilon L^{36\theta_0/5} + C \tag{2.16}$$

Finally for K_3, slightly different with K_2, we have

$$K_3 \leq C\varepsilon K^6 + C\varepsilon L^{6(1+\theta_0)/(3-\theta_0)} \leq C\varepsilon K^6 + C\varepsilon L^{36\theta_0/5} + C \tag{2.17}$$

Combine (2.13),(2.16),(2.17) and substitute into (2.9),we get

$$K \leq C\varepsilon L^{6\theta_0/5} + C \tag{2.18}$$

2.3 $L \le C$

Multiply (1.1) by $-\partial_{33}^2 u$ and integrate over $(t_1, t_2) \times \mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{|\partial_3 u|_2^2|(t_2)}{2} + |\nabla \partial_3 u|_{2,2}^2 = \iint \sum_{j,k=1}^{3} u_j \partial_j u_k \partial_{33}^2 u_k + \frac{|\partial_3 u|_2^2|(t_1)}{2} \\
= & -\iint \partial_3 u_j \partial_j u_k \partial_3 u_k + \frac{|\partial_3 u|_2^2|(t_1)}{2} \\
= & \iint \left[-\sum_{j,k=1}^{2} \partial_3 u_j \partial_j u_k \partial_3 u_k \right] \\
& + \iint \left[-\sum_{j=1}^{2} \partial_3 u_j \partial_j u_3 \partial_3 u_3 - \sum_{k=1}^{2} \partial_3 u_3 \partial_3 u_k \partial_3 u_k - \partial_3 u_3 \partial_3 u_3 \partial_3 u_3 \right] \\
& + \frac{|\partial_3 u|_2^2(t_1)}{2} \\
= & \iint \left[\partial_{3j}^2 u_j u_k \partial_3 u_k + \partial_3 u_j u_k \partial_{3j}^2 u_k \right] \\
& + \iint \left[\partial_{j3}^2 u_j u_3 \partial_3 u_3 + \partial_3 u_j u_3 \partial_{j3}^2 u_3 + 2 u_3 \partial_3 u_k \partial_{33}^2 u_k + 2 u_3 \partial_{33}^2 u_3 \partial_3 u_3 \right] \\
& + \frac{|\partial_3 u|_2^2|(t_1)}{2} \\
= & L_1 + L_2 + \frac{|\partial_3 u|_2^2|(t_1)}{2}
\end{aligned}
\tag{2.19}
$$

Now we estimate L_1, L_2. For L_1, it follows that

$$
\begin{aligned}
L_1 &\le C |\nabla \nabla_h u|_{2,2} |u_h|_{a,b} |\partial_3 u|_{c,d} \\
&\le C |\nabla \nabla_h u|_{2,2} |u|_{a_1,b_1}^{1-\theta} |u_h^3|_{a_2/3,b_2/3}^{\theta/3} |\nabla u|_{2,2}^{1-\sigma} |\partial_3 u|_{c_2,d_2}^{\sigma} \\
&\le C\varepsilon J K^{\theta} L^{\sigma} \\
&\le C\varepsilon L^{\sigma+\theta_0} K^{\theta} \\
&\le C\varepsilon L^2 + C\varepsilon K^{2\theta/(2-\theta_0-\sigma)}
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{cases}
\frac{1-\theta}{a_1} + \frac{\theta}{a_2} + \frac{1-\sigma}{2} + \frac{\sigma}{c_1} = \frac{1}{2} \\
\frac{1-\theta}{b_1} + \frac{\theta}{b_2} + \frac{1-\sigma}{2} + \frac{\sigma}{d_1} = \frac{1}{2}
\end{cases}
$$

In order to invoke Lemma 1.2, we shall have

$$
\begin{cases}
\frac{2}{a_1} + \frac{3}{b_1} \ge \frac{3}{2}, & 2 \le b_1 \le 6 \\
\frac{2}{a_2} + \frac{3}{b_2} \ge \frac{1}{2}, & 6 \le b_2 \le 18 \\
\frac{2}{c_1} + \frac{3}{d_1} \ge \frac{3}{2}, & 2 \le d_1 \le 6
\end{cases}
$$

Hence it is easy to check

$$\theta+\sigma\geq\frac{3}{2} \tag{2.21}$$

If the equality in (2.21) holds, then $1/2\leq\theta\leq 1$ and

$$\frac{2\theta}{2-\theta_0-\sigma}=\frac{2\theta}{\theta+1/2-\theta_0}$$

is a decreasing function of θ, attains its minimum $2/(3/2-\theta_0)$ at $\theta=1$. Thus (2.20) gives

$$L_1\leq C\varepsilon L^2+C\varepsilon K^{4/(3-2\theta_0)} \tag{2.22}$$

In this case,

$$\theta=1,\quad \sigma=\frac{1}{2}$$

and we can take

$$a_2=16,\ b_2=8;\quad c_1=\frac{8}{3},\ d_1=4$$

Now we are in a position to estimate L_2.

$$\begin{aligned} L_2 &\leq |\nabla\partial_3 u|_{2,2}|u_3|_{s,r}|\partial_3 u|_{2s/(s-2),2r/(r-2)}\\ &\leq C\varepsilon L^2 \end{aligned} \tag{2.23}$$

Combine (2.22),(2.23) and substitute into (2.19),we obtain

$$L\leq C\varepsilon K^{2/(3-2\theta_0)}+C\leq C\varepsilon L^{12\theta_0/[5(3-2\theta_0)]}+C=C\varepsilon L+C$$

Thus

$$L\leq C \tag{2.24}$$

Now (2.8),(2.24) give

$$J+L\leq C \tag{2.25}$$

The proof is completed.

Remark 2.1. *One can easily recover the estimates for J,K as in [4]. But the estimate for L is different in that we exhaust all the possibilities.*

Remark 2.2. *Through the proof, we use Hölder inequality, Young inequality freely, in which the only constraint comes from Lemma 1.2. And one can easily solve the system (2.5) if s,r are given. It is promising that the methods used here are useful in solving the one component regularity conjecture, as in Remark 1.2.*

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