Remarks on one component regularity for the Navier-Stokes equations IV

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Abstract

We establish sufficient conditions for the regularity of solutions of the Navier-Stokes system based on one component of the velocity. It is proved that if $u_3 \in L^{s,r}$ with

$$\frac{2}{s}+\frac{3}{r}\leq\frac{3}{4}+\frac{1}{2r}$$

and $10/3 < r \leq \infty$, then the solution is regular.

1. Introduction and the main result

We continue our study in [1] of the one component regularity problem for Leray-Hopf weak solutions

(u, p) of the three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} \partial_t u + u.\nabla u - \Delta u + \nabla p = 0; \\ div\, u = 0. \end{cases}$$

Indeed, the main result is as follows.

Theorem 1.1 Let u be a Leray-Hopf weak solution of (NS) with data $u_0 \in H^1(R^3)$, and $u_3 \in L^{s,r}$ with

$$\frac{2}{s} + \frac{3}{r} \le \frac{3}{4} + \frac{1}{2r}, \qquad \frac{10}{3} < r \le \infty$$

Then u is actually regular.

The proof is similar to that in [1], so we just estimate the key terms J_3, L_1.

2. Proof of the main theorem

As in [1], we need only to estimate J_3, L_1 more carefully. For the term J_3, we have:

$$\begin{aligned}
J_3 &\le C \iint |u_3|\, |\partial_3 u|\, |\nabla\nabla_h u| \\
&\le C \int |u_3|_r\, |\partial_3 u|_{2r/(r-2)}\, |\nabla\nabla_h u|_2 \\
&\le C \int |u_3|_r\, |\partial_3 u|_2^{1-3/r}\, |\partial_3 u|_6^{3/r}\, |\nabla\nabla_h u|_2 \\
&\le C \int |u_3|_r\, |\partial_3 u|_2^{1-2/s-3/r}\, |\partial_3 u|_2^{2/s}\, |\nabla\nabla_h u|_2^{2/r}\, |\nabla\partial_3 u|_2^{1/r}\, |\nabla\nabla_h u|_2 \\
&\le C\, |u_3|_{s,r}\, |\partial_3 u|_{2,2}^{1-2/s-3/r}\, |\partial_3 u|_{\infty,2}^{2/s}\, |\nabla\nabla_h u|_{2,2}^{2/r+1}\, |\nabla\partial_3 u|_{2,2}^{1/r} \\
&\le C\varepsilon J^{1+2/r} L^{2/s+1/r}
\end{aligned}$$

Thus

$$J \le C\varepsilon L^{(2/s+1/r)/(1-2/r)} + C$$

Now we estimate L_1 as follows:

$$\begin{aligned}
L_1 &\le C\iint |\nabla\nabla_h u|\,|u|\,|\partial_3 u| \\
&\le C\int |\nabla\nabla_h u|_2\,|u|_6\,|\partial_3 u|_2^{1/2}\,|\partial_3 u|_6^{1/2} \\
&\le C\int |\nabla\nabla_h u|_2\left[|\nabla_h u|_2^{2/3}\,|\partial_3 u|_2^{1/3}\right]|\partial_3 u|_2^{1/2}\left[|\nabla\nabla_h u|_2^{1/3}\,|\nabla\partial_3 u|_2^{1/6}\right] \\
&\le C\int |\nabla\nabla_h u|_2^{4/3}\,|\nabla_h u|_2^{2/3}\,|\partial_3 u|_2^{1/3}\,|\partial_3 u|_2^{1/2}\,|\nabla\partial_3 u|_2^{1/6} \\
&\le C\,|\nabla\nabla_h|_{2,2}^{4/3}\,|\nabla_h u|_{\infty,2}^{2/3}\,|\partial_3 u|_{\infty,2}^{1/3}\,|\partial_3 u|_{2,2}^{1/2}\,|\nabla\partial_3 u|_{2,2}^{1/6} \\
&\le C\varepsilon J^2 L^{1/2}
\end{aligned}$$

The estimate on J gives

$$L_1 \le C\varepsilon L^2 + C$$

For sufficiently small ε, we obtain $L \le C$, thus $J \le C$ also. The proof is complete.

3. Acknowledgement

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4. Reference

[1] Z. Zhang, Remarks on one component regularity for the Navier-Stokes equations, an exercise.