Remarks on one component regularity for the Navier-Stokes equations II

Zujin Zhang

October 1st, 2009

Department of Mathematics
Sun Yat-sen University
Guangzhou, P.R.China, 510275
Email: uia.china@gmail.com

Abstract

We establish sufficient conditions for the regularity of solutions of the Navier-Stokes system based on one component of the velocity. It is proved that if $u_3 \in L_t^s L_x^r$ with

$$\frac{2}{s} + \frac{3}{r} < \frac{5}{7}$$

then the solution is regular.

Key words: Navier-Stokes equations, regularity of solution, regularity criteria

1 Introduction and Main Result

We consider the following Cauchy problem for the incompressible Naiver-Stokes equations in $\mathbb{R}^3 \times (0,T)$:

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u - \triangle u + \nabla p = 0, \\ \text{div } u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with div $u_0 = 0$ in distributional sense is the initial velocity field.

The existence of a weak solution to (1.1) with $u_0 \in L^2(\mathbb{R}^3)$ is well known since the famous paper by Leray [6]. Its regularity and uniqueness remains still open. However, many criteria ensuring the smoothness of the solution are known. The classical Prodi-Serrin conditions (see [9, 10] and for $s = 3$ [3]) say that if the weak solution u additionally belongs to $L^{s,r}$ with $2/s+3/r \leq 1, 3 \leq r \leq \infty$, then the solution is as regular as the data allow and unique in the classof all weak solutions satisfying the energy inequality. Later on, criteria just for one velocity component appeared. The first result in this direction is due to Neustupa, Novotný and Penel [8] (see also Zhou's work [12]), where the authors showed that if $u_3 \in L^{s,r}$ with $2/s + 3/r \leq 1/2, 6 < s \leq \infty$, then the solution is smooth. Recently, three interesting improvements appeared. In [4], Kukavica and Ziane proved that if $u_3 \in L^{s,r}$ with $2/s+3/r \leq 5/8, 24/5 < r \leq \infty$, then the weak solution is regular. Next in [2], Cao and Titi use different method, instead of technical estimates they applied multiplicative Sobolev imbedding theorem and showed the smoothness under the assumption $u_3 \in L^{s,r}$ with $2/s + 3/r < 2/3 + 2/(3r), 7/2 < r \leq \infty$. And finally, in [13], Zhou proved regularity if $u_3 \in L^{s,r}$ with $2/s + 3/r \leq 3/4 + 1/(2r), 10/3 < r \leq \infty$. Here $L^{s,r}$ is defined by

$$|u|_{s,r} = ||u||_{L^{s,r}} = \begin{cases} \left(\int_0^T |u|_r^s(\tau)d\tau\right)^{1/s}, & \text{if } 1 \leq s < \infty, \\ \text{esssup}_{0\leq\tau\leq T}|u|_r(\tau), & \text{if } s = \infty. \end{cases}$$

where

$$|u|_r(\tau) = ||u||_{L^r}(\tau) = \begin{cases} \left(\int_{\mathbb{R}^3} |u(x,\tau)|^r dx\right)^{1/r}, & \text{if } 1 \leq r < \infty, \\ \text{esssup}_{x\in\mathbb{R}^3}|u(x,\tau)|, & \text{if } r = \infty. \end{cases}$$

The point is that $|u_\lambda|_{s,r} = |u|_{s,r}$ holds for all $\lambda > 0$ if and only if $2/s + 3/r = 1$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ and if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$ (See the dimensions table in [1]). Usually we say that the norm $|u|_{s,r}$ has the scaling dimension zero for $2/s + 3/r = 1$.

In this paper, we shall improve these and other known one-component regularity results. Our main result is

Theorem 1.1. *Let u be a Leary-Hopf weak solution of* (1.1) *with initial data $u_0 \in H^1(\mathbb{R}^3)$, and $u_3 \in L^{s,r}$ with*

$$\frac{2}{s} + \frac{3}{r} < \frac{5}{7}$$

then u actually is regular and unique on $(0,T)$.

Remark 1.1. *It improves the result in [4, 11], where in [11], the author of the present paper found in [4] the optimality of estimation for* J, K*, but not for* L *(*J, K, L *be defined below,* K *is slightly different, with* α *replaced by* 3*).*

Remark 1.2. *Since* $5/7 < 1$*, our result is not optimal in the sense of dimensional analysis. And it is remarked by Zhou in [12] that it is indeed a challenging problem to show regularity by adding Serrin's condition on only one velocity component. The key point is that one should lower the sum of powers in the convective term from* 3 *to less than* 2*. But only the terms with* u_3 *(or perhaps* ∇u *) are possible.*

Remark 1.3. *Note that due to the scaling properties of the Navier-Stokes equations, our result is better than the above-mentioned progress in one component regularity in some sense. Since we expect rather* $2/s + 3/r = const$*.*

In this paper, unless otherwise stated, we shall use C denote a generic constant depending only on the initial data, $|\nabla u|_2(t_1), |u_h^\alpha|_2^{1/\alpha}$ (t_1, u_h, α be defined below), and may change from line to line, and ε a sufficiently small constant which may differ by some power. Also, we will use Hölder inequality, Young inequality freely, sometimes without explanation. The reader may keep in mind that $\{a, b\}, \{c, d\}, \{p, q\}$ are Hölder conjugates, $\theta, \sigma, \tau \in [0, 1]$.

2 Preliminaries

We will use the following two results.

Lemma 2.1. *[12] Assume* $u \in L^{\infty,2}$ *and* $\nabla u \in L^{2,2}$ *on* $[0, T)$*, then* $u \in L^{a,b}$ *with* $2/a + 3/b \geq 3/2, 2 \leq b \leq 6$*. Moreover,*

$$|u|_{a,b} \leq C(p, q, T)|u|_{\infty,2}^{3/b-1/2}|\nabla u|_{2,2}^{3/2-3/b}$$

Lemma 2.2. *[5] Assume that* $u = (u_1, u_2, u_3) \in H^2(\mathbb{R}^3)$ *is smooth and divergence free, then*

$$\sum_{i,j=1}^{2} \int u_i \partial_i u_j \triangle_h u_j = \frac{1}{2} \sum_{i,j=1}^{2} \int \partial_i u_j \partial_i u_j \partial_3 u_3 - \int \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 + \int \partial_1 u_2 \partial_2 u_1 \partial_3 u_3$$

where $\triangle_h = \partial_{11}^2 + \partial_{22}^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ *is the two dimensional Laplacian.*

Before going to prove Theorem 1.1, we wish to recall the definition of Leary-Hopf weak solutions to (1.1) (see [7]) and a key observation from the theory of parabolic equations.

Definition 2.1. *A measurable vector u is called a Leary-Hopf weak solution to* (1.1)*, if u satisfies the following properties:*

(i) u is weakly continuous from $[0, T)$ *to* $L^2(\mathbb{R}^3)$*,*

(ii) u verifies (1.1) *in the sense of distributions,*

(iii) The energy inequality

$$|u|_2^2(t) + 2\int_0^t |\nabla u|_2^2(\tau)d\tau \le |u_0|^2, \quad \forall\ 0 \le t \le T.$$

By the definition and Sobolev imbedding, we know that the Leray-Hopf weak solutions satisfies

$$u \in L^{\infty,2} \cap L^{2,6}$$

thus

$$u_i u_j \in L^{\infty,1} \cap L^{1,3}, \quad 1 \le i, j \le 3$$

Interpolation between $L^{p,q}$ spaces gives

$$u_i u_j \in L^{p,q}$$

with p, q verify

$$\frac{2}{3p} + \frac{1}{q} = 1, \quad 1 < q < 3$$

Then the standard theory of parabolic equations (see Appendix D in [7]) implies

$$\nabla u \in L^{p,q}$$

We apply Fubini's Theorem and Sobolev imbedding to get for any fixed $\alpha \in (3/2, \infty)$,

$$u \in L^{\alpha}, \quad \text{a.e. } t \in [0, T].$$

Thus we can multiply (1.1) by u_h^{α} as in the next section.

3 Proof of Theorem 1.1

First note that as $u_0 \in H^1(\mathbb{R}^3)$, there is possibly a short time interval $(0, T^*)$ such that there exists a strong solution to the Navier-Stokes equations such that $u \in C^{\infty}((0, T^*) \times \mathbb{R}^3)$. Denote by $\mathcal{T}^*$ the supremum of of all such T^*, and assume $\mathcal{T}^* < T$. In what follows we will show that $|\nabla u|_2(t)$ remains bounded independently of $t \to \mathcal{T}^*_-$. The standard extension argument leads to contradiction.

We fix $\varepsilon > 0$ sufficiently small and $t_1 < \mathcal{T}^*$ such that

$$\int_{t_1}^{\mathcal{T}^*} |u_3|_r^s d\tau < \varepsilon, \quad \int_{t_1}^{\mathcal{T}^*} |\nabla u|_2^2 d\tau < \varepsilon$$

(therefore we need $s < \infty$). We take $t_2 \in (t_1, \mathcal{T}^*)$ arbitrary and our aim is to show that $|\nabla u|_2(t_2) \leq C$, here C may depend on the data, $|u_h^\alpha|_2(t_1)$ and $|\nabla u|_2(t_1)$, but is independent of t_2. Passing with t_2 to $\mathcal{T}^*$ we get the result.

We will work with

$$J(t_2) = |\nabla_h u|_{\infty,2} + |\nabla\nabla_h u|_{2,2}$$

$$L(t_2) = |\partial_3 u|_{\infty,2} + |\nabla\partial_3 u|_{2,2}$$

$$K(t_2) = |u_h^\alpha|_{\infty,2}^{1/\alpha} + |\nabla u_h^\alpha|_{2,2}^{1/\alpha}$$

where $\nabla_h = (\partial_1, \partial_2)$,$u_h = (u_1, u_2)$ and α is chosen so that if $2/s + 3/r = \theta_0 < 5/7$, then

$$\theta_0 = \frac{5}{2} \cdot \frac{4\alpha - 3}{14\alpha - 9}$$

Here and thereafter the integral is over $(t_1, t_2) \times \mathbb{R}^3$.

As a matter of fact, we need to show that

$$J(t_2) + L(t_2) \leq const < \infty$$

uniformly in t_2.

To this aim, we shall in the subsections below, estimate the terms mentioned above. For convenience, we write J, K, L for $J(t_2), K(t_2), L(t_2)$ respectively.

3.1 $J \leq C\varepsilon L^{\theta_0} + C$

Multiply (1.1) by $-\triangle_h u$, and integrate over $(t_1, t_2) \times \mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{|\nabla_h u|_2^2(t_2)}{2} + |\nabla\nabla_h u|_{2,2}^2 = \frac{|\nabla_h u|_2^2(t_1)}{2} + \iint u \cdot \nabla u \cdot \triangle_h u \\ = &\frac{|\nabla_h u|_2^2(t_1)}{2} + \iint \left[\sum_{i,j=1}^{2} u_j \partial_j u_i \triangle_h u_i + \sum_{j=1}^{3} u_j \partial_j u_3 \triangle_h u_3 + \sum_{i=1}^{2} u_3 \partial_3 u_i \triangle_h u_i \right] \\ = &\frac{|\nabla_h u|_2^2(t_1)}{2} + J_1 + J_2 + J_3 \qquad (3.2) \end{aligned}$$

We estimate the terms one by one.

$$
\begin{aligned}
J_1 &= \iint \sum_{i,j=1}^{2} u_j \partial_j u_i \triangle_h u_i \\
&= \iint \left[\frac{1}{2} \partial_3 u_3 \partial_i u_j \partial_i u_j - \partial_3 u_3 \partial_1 u_1 \partial_2 u_2 + \partial_3 u_3 \partial_1 u_2 \partial_2 u_1 \right] \\
&= \iint \left[u_3 \partial_{3i}^2 u_j \partial_i u_j + u_3 \partial_{31}^2 u_1 \partial_2 u_2 + u_3 \partial_1 u_1 \partial_{23}^2 u_2 \right] \\
&\quad + \iint \left[-u_3 \partial_{31}^2 u_2 \partial_2 u_1 - u_3 \partial_1 u_2 \partial_{32}^2 u_1 \right] \\
&\leq C |u_3|_{s,r} |\nabla_h u|_{2s/(s-2),2r/(r-2)} |\nabla \nabla_h u|_{2,2} \\
&\leq C \varepsilon J^2, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \iint \sum_{j=1}^{3} u_j \partial_j u_3 \triangle_h u_3 \\
&= \iint \sum_{j=1}^{3} \sum_{k=1}^{2} -\partial_k u_j \partial_j u_3 \partial_k u_3 \\
&= \iint \left[\partial_{jk}^2 u_j u_3 \partial_k u_3 + \partial_k u_j u_3 \partial_{jk}^2 u_3 \right] \\
&\leq C |u_3|_{s,r} |\nabla_h u|_{2s/(s-2),2r/(r-2)} |\nabla \nabla_h u|_{2,2} \\
&\leq C \varepsilon J^2, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \iint \sum_{k=1}^{2} u_3 \partial_3 u_k \triangle_h u_k \\
&\leq C |u_3|_{s,r} |\partial_3 u|_{a,b} |\triangle_h u|_{2,2} \\
&\leq |u_3|_{s,r} |\nabla u|_{2,2}^{1-\theta} |\partial_3 u|_{a_1,b_1}^{\theta} |\triangle_h u|_{2,2} \\
&\leq C \varepsilon L^{\theta} J \qquad (3.5)
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1}{s} + \frac{1}{a} = \frac{1}{2} \qquad \frac{1}{r} + \frac{1}{b} = \frac{1}{2} \\
\frac{1}{a} = \frac{1-\theta}{2} + \frac{\theta}{a_1} \\
\frac{1}{b} = \frac{1-\theta}{2} + \frac{\theta}{b_1} \\
\frac{2}{a_1} + \frac{3}{b_1} \geq \frac{3}{2} \qquad 2 \leq b_1 \leq 6
\end{cases}
$$

Thus we deduce easily that

$$\theta \geq \theta_0. \qquad (3.6)$$

If the equality in (3.6) holds, then

$$J_3 \le C\varepsilon L^{\theta_0} J \tag{3.7}$$

Combine (3.3),(3.4) and (3.7) together and substitute into (3.2), we obtain

$$J \le C\varepsilon L^{\theta_0} + C \tag{3.8}$$

3.2 $K \le C\varepsilon L^{9(\alpha-1)\theta_0/(5\alpha)} + C$

Multiply $(1.1)_h$ by $u_h^{2\alpha-1}$ and integrate over $(t_1,t_2)\times\mathbb{R}^3$, we obtain

$$\frac{|u_h^\alpha|_2^2(t_2)}{2\alpha} + \frac{2\alpha-1}{\alpha^2}|\nabla u_h^\alpha|_{2,2}^2 = (2\alpha-1)\iint \sum_{k=1}^{2} p u_k^{2\alpha-2}\partial_k u_k + \frac{|u_h^\alpha|_2^2(t_1)}{2\alpha} \tag{3.9}$$

By the well-known identity

$$-\triangle p = \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i,$$

we can write $p = p_1 + p_2 + p_3$, where the p_i's solve

$$-\triangle p_1 = \sum_{i,j=1}^{2} \partial_i u_j \partial_j u_i$$

$$-\triangle p_2 = 2\sum_{i=1}^{2} \partial_i(u_3\partial_3 u_i) + \partial_3(u_3\partial_3 u_3)$$

$$-\triangle p_3 = u_3\partial_{33}^2 u_3$$

Now (3.9) gives

$$\begin{aligned}\frac{|u_h^\alpha|_2^2(t_2)}{2\alpha} + \frac{2\alpha-1}{\alpha^2}|\nabla u_h^\alpha|_{2,2}^2 &= \sum_{l=1}^{3}(2\alpha-1)\iint \sum_{k=1}^{2} p_l u_k^{2\alpha-2}\partial_k u_k + \frac{|u_h^\alpha|_2^2(t_1)}{2\alpha} \\ &= \sum_{l=1}^{3} K_l + \frac{|u_h^\alpha|_2^2(t_1)}{2\alpha} \end{aligned}\tag{3.10}$$

We will estimate the K_l's differently. For K_1,

$$
\begin{aligned}
K_1 &= (2\alpha-1)\iint\sum_{k=1}^{2} p_1 u_k^{2\alpha-2}\partial_k u_k \\
&\le C|p_1|_{a,b}|u_h^{2\alpha-2}|_{c,d}|\nabla_h u|_{p,q} \\
&\le C|\triangle p_1|_{a,1/(1/b+2/3)}|u_h|^{2\alpha-2}_{(2\alpha-2)c,(2\alpha-2)d}|\nabla_h u|_{p,q} \\
&\le |\nabla_h u|^2_{2a,2/(1/b+2/3)}|u_h|^{2\alpha-2}_{(2\alpha-2)c,(2\alpha-2)d}|\nabla_h u|_{p,q} \\
&\le C|\nabla u|^{2(1-\theta)}_{2,2}|\nabla_h u|^{2\theta}_{a_1,b_1}|u|^{(2\alpha-2)(1-\sigma)}_{c_1,d_1}|u_h^\alpha|^{(2\alpha-2)\sigma/\alpha}_{c_2/\alpha,d_2/\alpha}|\nabla u|^{1-\tau}_{2,2}|\nabla_h u|^{\tau}_{p_1,q_1} \\
&\le C\varepsilon J^{2\theta}K^{(2\alpha-2)\sigma}J^{\tau} \\
&\le C\varepsilon J^{\tau+2\theta}K^{(2\alpha-2)\sigma} \\
&\le C\varepsilon K^{2\alpha}+C\varepsilon J^{\alpha(\tau+2\theta)/[\alpha-(\alpha-1)\sigma]} \\
&\le C\varepsilon K^{2\alpha}+C\varepsilon L^{\alpha\theta_0(\tau+2\theta)/[\alpha-(\alpha-1)\sigma]} \qquad (3.11)
\end{aligned}
$$

In the sequel, we use Hölder inequality, the decomposition of p, Lemma 2.1, and finally $J\le C\varepsilon L^{\theta_0}+C$ just proved in the previous subsection. More precisely, we shall have

$$
\begin{cases}
\frac{1}{a}+\frac{1}{c}+\frac{1}{p}=1 & \frac{1}{b}+\frac{1}{d}+\frac{1}{q}=1 \\
\frac{1}{2a}=\frac{1-\theta}{2}+\frac{\theta}{a_1} & \frac{1}{2}\left(\frac{1}{b}+\frac{2}{3}\right)=\frac{1-\theta}{2}+\frac{\theta}{b_1} \\
\frac{1}{(2\alpha-2)c}=\frac{1-\sigma}{c_1}+\frac{\sigma}{c_2} & \frac{1}{(2\alpha-2)d}=\frac{1-\sigma}{d_1}+\frac{\sigma}{d_2} \\
\frac{1}{p}=\frac{1-\tau}{2}+\frac{\tau}{p_1} & \frac{1}{q}=\frac{1-\tau}{2}+\frac{\tau}{q_1}
\end{cases}
$$

and

$$
\begin{cases}
\frac{2}{a_1}+\frac{2}{b_1}\ge\frac{3}{2} & 2\le b_1\le 6 \\
\frac{2}{c_1}+\frac{2}{d_1}\ge\frac{3}{2} & 2\le d_1\le 6 \\
\frac{2}{c_2}+\frac{2}{d_2}\ge\frac{3}{2\alpha} & 2\alpha\le d_2\le 6\alpha \\
\frac{2}{p_1}+\frac{3}{q_1}\ge\frac{3}{2} & 2\le q_1\le 6
\end{cases}
$$

It follows that

$$
\begin{cases}
1-\theta+\frac{2\theta}{a_1}+\frac{(2\alpha-2)(1-\sigma)}{c_1}+\frac{(2\alpha-2)\sigma}{c_2}+\frac{1-\tau}{2}+\frac{\tau}{p_1}=1 \\
1-\theta+\frac{2\theta}{b_1}+\frac{(2\alpha-2)(1-\sigma)}{d_1}+\frac{(2\alpha-2)\sigma}{d_2}+\frac{1-\tau}{2}+\frac{\tau}{q_1}=1
\end{cases}
$$

Hence

$$
\tau+2\theta\ge 3\alpha-\frac{5}{2}-\frac{3}{\alpha}(\alpha-1)^2\sigma \qquad (3.12)
$$

If the equality in (3.12) holds, then

$$
\frac{\alpha(3\alpha-11/2)}{3(\alpha-1)^2}\le\sigma\le 1
$$

and

$$\frac{\alpha\theta_0(\tau+2\theta)}{\alpha-(\alpha-1)\theta}=\alpha\theta_0\frac{3\alpha-5/2-3(\alpha-1)^2\sigma/\alpha}{\alpha-(\alpha-1)\sigma}$$

is an increasing function of σ, attains its minimum $18(\alpha-1)\theta_0/5$ at

$$\frac{\alpha(3\alpha-11/2)}{3(\alpha-1)^2}$$

Thus (3.11) gives

$$K_1\leq C\varepsilon K^{2\alpha}+C\varepsilon L^{18(\alpha-1)\theta_0/5} \tag{3.13}$$

For K_2,

$$\begin{aligned}
K_2 &= (2\alpha-1)\iint\sum_{k=1}^{2}p_2u_k^{2\alpha-2}\partial_k u_k\\
&\leq C|p_2|_{a,b}|u_h^{2\alpha-2}|_{c,d}|\nabla_h u|_{p,q}\\
&\leq C|\nabla p_2|_{a,1/(1/b+1/3)}|u_h|^{2\alpha-2}_{(2\alpha-2)c,(2\alpha-2)d}|\nabla_h u|_{p,q}\\
&\leq C|u_3\partial_3 u|_{a,1/(1/b+1/3)}|u_h|^{2\alpha-2}_{(2\alpha-2)c,(2\alpha-2)d}|\nabla_h u|_{p,q}\\
&\leq C|u_3|_{s,r}|\partial_3 u|_{a_1,b_1}|u|^{(2\alpha-2)(1-\theta)}_{c_1,d_1}|u_h^\alpha|^{(2\alpha-2)\theta/\alpha}_{c_2/\alpha,d_2/\alpha}|\nabla_h u|_{p,q}\\
&\leq C\varepsilon LK^{(2\alpha-2)\theta}J\\
&\leq C\varepsilon L^{1+\theta_0}K^{(2\alpha-2)\theta}\\
&\leq C\varepsilon K^{2\alpha}+C\varepsilon L^{\alpha(1+\theta_0)/[\alpha-(\alpha-1)\theta]} \qquad (3.14)
\end{aligned}$$

In the sequel, we shall have

$$\begin{cases}\frac{1}{s}+\frac{1}{a_1}+\frac{(2\alpha-2)(1-\theta)}{c_1}+\frac{(2\alpha-2)\theta}{c_2}+\frac{1}{p}=1\\ \frac{1}{r}+\left(\frac{1}{b_1}-\frac{1}{3}\right)+\frac{(2\alpha-2)(1-\theta)}{d_1}+\frac{(2\alpha-2)\theta}{d_2}+\frac{1}{q}=1\end{cases}$$

and it is easily deduced that

$$\theta\geq\frac{\alpha(\theta_0+3\alpha-6)}{3(\alpha-1)^2} \tag{3.15}$$

if we invoke Lemma 2.1 to verify (3.14). However, if the equality in (3.15) holds,we have

$$K_2\leq C\varepsilon K^{2\alpha}+C\varepsilon L^{3(\alpha-1)(1+\theta_0)/(3-\theta_0)}\leq C\varepsilon K^{2\alpha}+C\varepsilon L^{18(\alpha-1)\theta_0/5}+C \tag{3.16}$$

Finally for K_3,

$$\begin{aligned}
K_3 &= (2\alpha-1)\iint \sum_{k=1}^{2} p_3 u_k^{2\alpha-2}\partial_k u_k \\
&\leq C|p_3|_{a,b}|u_h^{2\alpha-2}|_{c,d}|\nabla_h u|_{p,q} \\
&\leq C|\triangle p_3|_{a,1/(1/b+2/3)}|u_h|^{2\alpha-2}_{(2\alpha-2)c,(2\alpha-2)d}|\nabla_h u|_{p,q} \\
&\leq C|u_3|_{s,r}|\nabla\partial_3 u|_{2,2}|u|^{(2\alpha-2)(1-\sigma)}_{c_1,d_1}|u_h^{\alpha}|^{(2\alpha-2)\sigma/\alpha}_{c_2/\alpha,d_2/\alpha}|\nabla_h u|_{p,q} \\
&\leq C\varepsilon L K^{(2\alpha-2)\sigma} J \\
&\leq C\varepsilon L^{1+\theta_0} K^{(2\alpha-2)\sigma} \\
&\leq C\varepsilon K^{2\alpha} + L^{\alpha(1+\theta_0)/[\alpha-(\alpha-1)\sigma]} \qquad (3.17)
\end{aligned}$$

where as done several times before, we shall have

$$\begin{cases} \frac{1}{s}+\frac{1}{2}+\frac{(2\alpha-2)(1-\sigma)}{c_1}+\frac{(2\alpha-2)\sigma}{c_2}+\frac{1}{p}=1 \\ \frac{1}{r}+\frac{1}{2}-\frac{2}{3}+\frac{(2\alpha-2)(1-\sigma)}{d_1}+\frac{(2\alpha-2)\sigma}{d_2}+\frac{1}{q}=1 \end{cases}$$

and

$$\theta \geq \frac{\alpha(\theta_0+3\alpha-6)}{3(\alpha-1)^2}.$$

Hence

$$K_3 \leq C\varepsilon K^{2\alpha} + C\varepsilon L^{3(\alpha-1)(1+\theta_0)/(3-\theta_0)} \leq C\varepsilon K^{2\alpha} + C\varepsilon L^{18(\alpha-1)\theta_0/5} + C \qquad (3.18)$$

Combine (3.13),(3.16),(3.18) and substitute into (3.10),we get

$$K \leq C\varepsilon L^{9(\alpha-1)\theta_0/(5\alpha)} + C \qquad (3.19)$$

3.3 $L \le C$

Multiply (1.1) by $-\partial_{33}^2 u$ and integrate over $(t_1, t_2) \times \mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{|\partial_3 u|_2^2(t_2)}{2} + |\nabla \partial_3 u|_{2,2}^2 = \iint \sum_{j,k=1}^{3} u_j \partial_j u_k \partial_{33}^2 u_k + \frac{|\partial_3 u|_2^2(t_1)}{2} \\
= & -\iint \partial_3 u_j \partial_j u_k \partial_3 u_k + \frac{|\partial_3 u|_2^2(t_1)}{2} \\
= & \iint \left[-\sum_{j,k=1}^{2} \partial_3 u_j \partial_j u_k \partial_3 u_k \right] \\
& + \iint \left[-\sum_{j=1}^{2} \partial_3 u_j \partial_j u_3 \partial_3 u_3 - \sum_{k=1}^{2} \partial_3 u_3 \partial_3 u_k \partial_3 u_k - \partial_3 u_3 \partial_3 u_3 \partial_3 u_3 \right] \\
& + \frac{|\partial_3 u|_2^2(t_1)}{2} \\
= & \iint \left[\partial_{3j}^2 u_j u_k \partial_3 u_k + \partial_3 u_j u_k \partial_{3j}^2 u_k \right] \\
& + \iint \left[\partial_{j3}^2 u_j u_3 \partial_3 u_3 + \partial_3 u_j u_3 \partial_{j3}^2 u_3 + 2 u_3 \partial_3 u_k \partial_{33}^2 u_k + 2 u_3 \partial_{33}^2 u_3 \partial_3 u_3 \right] \\
& + \frac{|\partial_3 u|_2^2(t_1)}{2} \\
= & L_1 + L_2 + \frac{|\partial_3 u|_2^2(t_1)}{2} \qquad (3.20)
\end{aligned}
$$

Now we estimate L_1, L_2. For L_1, it follows that

$$
\begin{aligned}
L_1 &\le C |\nabla \nabla_h u|_{2,2} |u_h|_{a,b} |\partial_3 u|_{c,d} \\
&\le C |\nabla \nabla_h u|_{2,2} |u|_{a_1,b_1}^{1-\theta} |u_h^\alpha|_{a_2/\alpha, b_2/\alpha}^{\theta/\alpha} |\nabla u|_{2,2}^{1-\sigma} |\partial_3 u|_{c_2,d_2}^{\sigma} \\
&\le C \varepsilon J K^\theta L^\sigma \\
&\le C \varepsilon L^{\sigma+\theta_0} K^\theta \\
&\le C \varepsilon L^2 + C \varepsilon K^{2\theta/(2-\theta_0-\sigma)} \qquad (3.21)
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1-\theta}{a_1} + \frac{\theta}{a_2} + \frac{1-\sigma}{2} + \frac{\sigma}{c_1} = \frac{1}{2} \\
\frac{1-\theta}{b_1} + \frac{\theta}{b_2} + \frac{1-\sigma}{2} + \frac{\sigma}{d_1} = \frac{1}{2}
\end{cases}
$$

In order to invoke Lemma 2.1, we shall have

$$
\begin{cases}
\frac{2}{a_1} + \frac{3}{b_1} \ge \frac{3}{2} & 2 \le b_1 \le 6 \\
\frac{2}{a_2} + \frac{3}{b_2} \ge \frac{3}{2\alpha} & 2\alpha \le b_2 \le 6\alpha \\
\frac{2}{c_1} + \frac{3}{d_1} \ge \frac{3}{2} & 2 \le d_1 \le 6
\end{cases}
$$

Hence it is easy to check

$$\frac{3(\alpha-1)}{2\alpha}\theta+\sigma\geq\frac{3}{2} \tag{3.22}$$

If the equality in (3.22) holds, then $\alpha/[3(\alpha-1)]\leq\theta\leq 1$ and

$$\frac{2\theta}{2-\theta_0-\sigma}=\frac{4\alpha\theta}{(3\alpha-3)\theta+\alpha-2\alpha\theta_0}$$

is a decreasing function of θ, attains its minimum $4\alpha/(4\alpha-3-2\alpha\theta_0)$ at $\theta=1$. Hence (3.21) gives

$$L_1\leq C\varepsilon L^2+C\varepsilon K^{4\alpha/(4\alpha-3-2\alpha\theta_0)} \tag{3.23}$$

We are now in a position to estimate L_2.

$$\begin{aligned} L_2 &\leq |\nabla\partial_3 u|_{2,2}|u_3|_{s,r}|\partial_3 u|_{2s/(s-2),2r/(r-2)} \\ &\leq C\varepsilon L^2 \end{aligned} \tag{3.24}$$

Now we conclude. Combine (3.23),(3.24) and (3.19), we obtain

$$\begin{aligned} L &\leq C\varepsilon K^{2\alpha/(4\alpha-3-2\alpha\theta_0)}+C \\ &\leq C\varepsilon L^{18(\alpha-1)\theta_0/[5(4\alpha-3-2\alpha\theta_0)]}+C \\ &\leq C\varepsilon L+C \end{aligned} \tag{3.25}$$

Thus, (3.8) and (3.25) together imply

$$J+L\leq C$$

as required.

Remark 3.1. *Through the proof, we use Hölder inequality, Young inequality freely, in which the only constraint comes from Lemma 2.1. One can easily calculate* $a,b;c,d;p,q$ *and* θ,σ,τ *if* s,r *are given.*

References

[1] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the navier-stokes equations. *Comm. Pure Appl. Math.*, 35:771–831, 1982.

[2] C. Cao and E. S. Titi. Regularity criteria for the three dimensional navier-stokes equations. *Indiana Univ. Math. J.*, 57(6):2643–2661, 2008.

[3] L. Escauriaza and V.*Š*ver*á*k G.Seregin. Backward uniqueness for parabolic equations. *Arch. Ration. Mech. Anal.*, 169(2):147–157, 2003.

[4] I. Kukavica and M. Ziane. One component regularity for the navier-stokes equations. *Nonlinearity*, 19(2), 2006.

[5] I. Kukavica and M. Ziane. Navier-stokes equations with regularity in one direction. *J. Math. Phys.*, 48(6):10 pp, 2007.

[6] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, (63):193–248, 1934.

[7] P. Lions. *Mathematical topics in fliud mechanics.* The Clarendon Press Oxford University Press, 1998,10.

[8] J. Neustupa, A. Novotn*ý*, and P. Penel. An interior regularity of a weak solution to the navier-stokes equations in dependence on one component of velocity. *Topics in mathematical fluid mechanics*, 10:163–183, 2002.

[9] G. Prodi. Un teorema di unicit*à* per el equazioni di navier-stokes. *Ann. Mat. Pura Appl.*, IV(48):173–182, 1959.

[10] J. Serrin. The initial value problem for the navier-stokes equations. *Nonlinear Problems*, pages 69–98, 1963.

[11] Z. Zhang. Remarks on one component regularity for the navier-stokes equations. *submitted*, 2009.

[12] Y. Zhou. A new regularity criterion for weak solutions to the navier-stokes equations. 84:1496–1514, 2005.

[13] Y. Zhou. On the regularity to the solutions of the navier-stokes equations via one velocity component. *submitted*, 2009.